Math 776 – Spring 2011 – Homework 8

Problems 1–3 are taken from Stein and Shakarchi, Real Analysis, Princeton University Press (2005) and are due on Monday April 11.

1. (p. 146, Exercise 4) Prove that if \( f \) is integrable on \( \mathbb{R} \), and \( f \) is not identically zero, then there is a constant \( c > 0 \) such that for all \( x \) with \( |x| \geq 1 \), \( f^*(x) \geq c|x|^{-1} \). Conclude that \( f^* \) is not integrable on \( \mathbb{R} \). [Hint: Use the fact that for some interval \( I \) centered at the origin, \( \int_I |f| > 0 \).]

2. (p. 146, Exercise 5) Consider the function on \( \mathbb{R} \) defined by
   \[
   f(x) = \begin{cases} 
   1/\sqrt{|x|(|\log 1/|x||)^2} & \text{if } |x| \leq 1/2 \\
   0 & \text{otherwise}
   \end{cases}
   \]
   (a) Verify that \( f \) is integrable on \( \mathbb{R} \).
   (b) Show that for some \( c > 0 \) and all \( |x| \leq 1/2 \),
   \[
   f^*(x) \geq c|x|(|\log 1/|x||)^2.
   \]
   Conclude that \( f^* \) is not locally integrable.

3. (p. 152, Problem 2) Suppose that \( I_1, I_2, \ldots, I_N \) is a given finite collection of open intervals in \( \mathbb{R} \). then there are two finite sub-collections \( I'_1, I'_2, \ldots, I'_K \) and \( I''_1, I''_2, \ldots, I''_L \) such that each sub-collection consists of mutually disjoint intervals and
   \[
   \bigcup_{j=1}^N I_j = \bigcup_{k=1}^K I'_k \cup \bigcup_{l=1}^L I''_l.
   \]
   Conclude from this that given a finite collection of open intervals \( \{I_j\}_{j=1}^N \), we can find a disjoint sub-collection \( \{I_{jk}\}_{k=1}^K \) such that
   \[
   m\left(\bigcup_{j=1}^N I_j\right) \leq 2 \sum_{k=1}^K m(I_{jk}).
   \]
   [Hint: Choose \( I'_1 \) to be an interval whose left endpoint is as far left as possible. Discard all intervals contained in \( I'_1 \). If the remaining intervals are disjoint from \( I'_1 \), select again an interval as far to the left as possible and call it \( I'_2 \). Otherwise choose an interval that intersects \( I'_1 \), but reaches out to the right as far as possible, and call this interval \( I''_1 \). Repeat this procedure.]
