Math 776 – Spring 2011 – Homework 8

Problems 1–3 are taken from Stein and Shakarchi, *Real Analysis*, Princeton University Press (2005) and are due on Monday April 11.

1. (p. 146, Exercise 4) Prove that if f is integrable on $\mathbf{R}$, and f is not identically zero, then there is a constant $c > 0$ such that for all x with $|x| \geq 1$, $f^*(x) \geq c|x|^{-1}$. Conclude that f^* is not integrable on $\mathbf{R}$. [Hint: Use the fact that for some interval I centered at the origin, $\int_I |f| > 0$].

2. (p. 146, Exercise 5) Consider the function on $\mathbf{R}$ defined by

$$f(x) = \begin{cases} 1/[|x|(\log 1/|x|)^2] & \text{if } |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is integrable on $\mathbf{R}$.

(b) Show that for some $c > 0$ and all $|x| \leq 1/2$,

$$f^*(x) \geq \frac{c}{|x|(\log 1/|x|)}.$$

Conclude that f^* is not locally integrable.

3. (p. 152, Problem 2) Suppose that $I_1, I_2, \ldots, I_N$ is a given finite collection of open intervals in $\mathbf{R}$. then there are two finite sub-collections $I'_1, I'_2, \ldots, I'_K$ and $I''_1, I''_2, \ldots, I''_L$ such that each sub-collection consists of mutually disjoint intervals and

$$\bigcup_{j=1}^{N} I_j = \bigcup_{k=1}^{K} I'_k \cup \bigcup_{l=1}^{L} I''_l.$$

Conclude from this that given a finite collection of open intervals $\{I_j\}_{j=1}^N$, we can find a disjoint sub-collection $\{I_{j_k}\}_{k=1}^K$ such that

$$m\left(\bigcup_{j=1}^{N} I_j\right) \leq 2\sum_{k=1}^{K} m(I_{j_k}).$$

[Hint: Choose I'_1 to be an interval whose left endpoint is as far left as possible. Discard all intervals contained in I'_1. If the remaining intervals are disjoint from I'_1, select again an interval as far to the left as possible and call it I'_2. Otherwise choose an interval that intersects I'_1, but reaches out to the right as far as possible, and call this interval I''_1. Repeat this procedure.]

4. Royden and Fitzpatrick, p. 115, Problem 13.

5. Royden and Fitzpatrick, p. 115, Problem 15.