Zak Transform and the Balian–Low Theorem.

The Zak transform is an effective tool for analyzing Gabor systems when $\alpha\beta = 1$, though it is also very useful when $\alpha\beta$ is rational. In this lecture we will show how this tool can be used to prove two important theorems about Gabor systems.

A. Basic Properties of the Zak Transform.

Definition 1 *Given $\alpha > 0$ the* Zak transform *is defined formally as*

$$Z_\alpha f(x,\omega) = \sum_k f(x-\alpha k)\, e^{2\pi i\alpha k x}.$$

Remarks. 1. $Z_\alpha f(x,\omega)$ is clearly defined, for example, for functions $f(x)$ that are continuous and compactly supported, in which case the sum defining $Z_\alpha f$ is finite for each fixed x. In general, for each fixed x, $Z_\alpha f(x,\omega)$ is the $1/\alpha$–periodic Fourier series whose coefficients are given by $\{f(x-\alpha k)\}_{k\in\mathbf{Z}}$.
2. Again reasoning formally, $Z_\alpha f$ satisfies the *quasi–periodicity* relations, namely

$$Z_\alpha f(x+\alpha,\omega) = e^{2\pi i\alpha\omega}\, Z_\alpha f(x,\omega) \quad\text{and}\quad Z_\alpha f(x,\omega+1/\alpha) = Z_\alpha f(x,\omega).$$

Hence $Z_\alpha f(x,\omega)$ is completely determined by its values on the cube $Q = [0,\alpha]\times[0,1/\alpha]$ or any cube of the form $Q + (n\alpha, m/\alpha)$, m, $n\in\mathbf{Z}$.

Lemma 1 (a) *If $f \in W(\mathbf{R})$, then $Z_\alpha f(x,\omega)$ converges uniformly on $\mathbf{R}^2$ to a bounded function. Moreover, we have that $\|Z_\alpha f\|_\infty \le \|f\|_{W,\alpha}$.*

(b) *If $f\in W(\mathbf{R})$ is also continuous on $\mathbf{R}$, then $Z_\alpha f(x,\omega)$ converges uniformly on $\mathbf{R}^2$ to a bounded function continuous on all of $\mathbf{R}^2$.*

Examples. 1. If $f = \mathbf{1}_{[0,\alpha)}$ then $Z_\alpha f(x,\omega) = e^{2\pi i n\alpha\omega}$ where $n = n(x)$ is the unique integer such that $x\in[\alpha n, \alpha(n+1))$, or in other words, n is the greatest integer less than or equal to x/α. In particular, when $x\in[0,\alpha)$, $n=0$ so that on Q, $Z_\alpha f(x,\omega) = 1$.
2. If $\alpha = 1$ and $\varphi(x) = e^{-\pi x^2}$ then $Z_\alpha f(x,\omega) = \sum_k e^{-\pi(x-k)^2}\, e^{2\pi i k\omega}$ and it is easy to see that $Z_\alpha f(x,\omega)$ is continuous on $\mathbf{R}^2$ and $Z_\alpha f(1/2,1/2) = 0$.

Lemma 2 *(Inversion, Isometry and the Fourier Transform) For f sufficiently regular,*

(a) $f(x) = \alpha \displaystyle\int_0^{1/\alpha} Z_\alpha f(x,\omega)\, d\omega.$

(b) $\widehat{f}(\omega) = \displaystyle\int_0^{\alpha} Z_\alpha f(x,\omega)\, e^{2\pi i x\omega}\, dx.$

(c) $\|Z_\alpha f\|^2_{L^2(Q)} = \dfrac{1}{\alpha}\|f\|_2^2.$

(d) $Z_\alpha f(x,\omega) = \dfrac{1}{\alpha}\, e^{2\pi i x\omega}\, Z_{1/\alpha}\widehat{f}(\omega,-x).$

Lemma 3 *Suppose $\alpha\beta = 1$ and k, $n \in \mathbf{Z}$. Then*

$$Z_\alpha(T_{\alpha k}\, M_{\beta n} f)(x\,\omega) = e^{2\pi i n x/\alpha}\, e^{-2\pi i k\alpha\omega}\, Z_\alpha f(x,\omega).$$

Example. $Z_\alpha(T_{\alpha k}\, M_{\beta n}\mathbf{1}_{[0,\alpha)})(x\,\omega) = e^{2\pi i n x/\alpha}\, e^{-2\pi i k\alpha\omega}$ for $(x,\omega) \in Q$. Recall that

$$\{\alpha^{-1/2} T_{\alpha k}\, M_{\beta n}\mathbf{1}_{[0,\alpha)}\}_{k,n\in\mathbf{Z}}$$

is an orthonormal basis for $L^2(\mathbf{R})$ and note that $\{e^{2\pi i n x/\alpha}\, e^{-2\pi i k\alpha\omega}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$. Hence Z_α maps one orthonormal basis onto another (up to the constant $\alpha^{-1/2}$) and this identifies Z_α as a unitary operator and provides another proof of the previous Lemma.

B. The Zak Transform and Gabor frames.

Theorem 1 *Let g, $\gamma \in L^2(\mathbf{R})$ and let $\alpha\beta = 1$. Then*

$$Z_\alpha\, S_{g,\gamma} f = \alpha\, \overline{Z_\alpha g}\, \mathbf{Z}_\alpha\gamma\, Z_\alpha f.$$

Remarks. 1. The above theorem says that whenever $\alpha\beta = 1$ the Gabor frame operator reduces to a multiplication operator in the "Zak transform domain."
2. In this case, computing the inverse frame operator $S_{g,g}^{-1}$ for the Gabor frame $\mathcal{G}(g,\alpha,1/\alpha)$ is trivial namely

$$Z_\alpha\, S_{g,g}^{-1} f = \alpha^{-1}\,(\overline{Z_\alpha g}\, \mathbf{Z}_\alpha\gamma)^{-1}\, Z_\alpha f.$$

It is also possible to make statements about the frame and basis properties of Gabor systems at the "critical density" $\alpha\beta = 1$.

Corollary 1 *Let g, $\gamma \in L^2(\mathbf{R})$ and let $\alpha\beta = 1$. Then*

(a) *$S_{g,\gamma}$ is bounded if and only if $\overline{Z_\alpha g}\, \mathbf{Z}_\alpha\gamma$ is bounded on Q.*

(b) *$\mathcal{G}(g,\alpha,\beta)$ is a frame (and hence a Reisz basis) for $L^2(\mathbf{R})$ if and only if there exist constants a, $b > 0$ such that $a \le |Z_\alpha g(x,\omega)| \le b$ on Q.*

(c) *$\mathcal{G}(g,\alpha,\beta)$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|Z_\alpha g(x,\omega)| = \alpha^{-1/2}$ on Q.*

Corollary 2 *With $\varphi(x) = e^{-\pi x^2}$, $\mathcal{G}(\varphi,\alpha,1/\alpha)$ is not a frame for any $\alpha > 0$.*

C. The Balian-Low Theorem.

Lemma 4 *If $Z_\alpha f(x,\omega)$ is continuous on $\mathbf{R}^2$, then it has a zero in Q.*

Theorem 2 *If either g or $\widehat{g}$ are continuous functions in $W(\mathbf{R})$, then $\mathcal{G}(g, \alpha, 1/\alpha)$ cannot be a frame (and hence not a Riesz basis) for $L^2(\mathbf{R})$.*

Remark. 1. The above theorem says that in order to have a Gabor Riesz basis for $L^2(\mathbf{R})$ then the window g cannot be "well localized" in both time and frequency. Here "well localized" is taken to mean that g and $\widehat{g}$ must both be continuous and in $W(\mathbf{R})$.

2. A classical result in this direction that bears a closer relation to the classical uncertainty principle is referred to as the Balian–Low Theorem. This result says that if the window g generates a Gabor Riesz basis then it must *maximize* the uncertainty principle, that is, the variance of either g or of $\widehat{g}$ must be infinite.

Theorem 3 *(Balian–Low Theorem) If $\mathcal{G}(g, \alpha, 1/\alpha)$ is a frame for $L^2(\mathbf{R})$ then*

$$\|x\, g(x)\|_2 \|\omega \widehat{g}(\omega)\|_2 = \infty.$$