Gabor Systems: Duality and Density.

A. The Janssen representation.

Corollary 1 *Given g, $\gamma \in W(\mathbf{R})$, the operator $S_{g,\gamma}$ defined by*

$$S_{g,\gamma} f = \sum_{k,n} \langle f, T_{\alpha k} M_{\beta n} g \rangle \, T_{\alpha k} M_{\beta n} \gamma$$

is given by

$$S_{g,\gamma} f(x) = \frac{1}{\beta} \sum_{n} G_n(x) \, f(x - n/\beta)$$

where

$$G_n(x) = \sum_k \overline{g(x - n/\beta - \alpha k)} \, \gamma(x - \alpha k) = \sum_k T_{\alpha k}(T_{n/\beta} \overline{g} \cdot \gamma)(x).$$

Remark. Of course $S_{g,\gamma}$ is very similar to the frame operator for the Gabor system $\mathcal{G}(g, \alpha, \beta)$ which in this notation would be $S_{g,g}$. Also note that as written $S^*_{g,\gamma} = S_{\gamma,g}$ so that $S_{g,\gamma}$ is not self-adjoint However, if γ is dual to g, then $S_{g,\gamma} = S^*_{g,\gamma} = I$.

Theorem 1 *(Janssen's Representation) For g, $\gamma \in L^2(\mathbf{R})$ sufficiently regular,*

$$S_{g,\gamma} = (\alpha\beta)^{-1} \sum_{k,n} \langle \gamma, T_{k/\beta} M_{n/\alpha} g \rangle \, T_{k/\beta} M_{n/\alpha} = (\alpha\beta)^{-1} \sum_{k,n} \langle \gamma, M_{n/\alpha} T_{k/\beta} g \rangle \, M_{n/\alpha} T_{k/\beta}.$$

The proof of this Theorem follows immediately from the following Lemma.

Lemma 1 *For g, $\gamma \in L^2(\mathbf{R})$ sufficiently regular and for each n,*

$$G_n(x) = \alpha^{-1} \sum_k \langle \gamma, M_{k/\alpha} T_{n/\beta} g \rangle \, e^{2\pi i k x/\alpha}$$

where G_n is defined above.

Theorem 2 *(Wexler-Raz Biorthogonality Relations) If g, $\gamma \in L^2(\mathbf{R})$ are sufficiently regular then the following are equivalent.*

(a) $S_{g,\gamma} = S_{\gamma,g} = I$. *In other words, γ is dual to g.*

(b) $(\alpha\beta)^{-1} \langle \gamma, M_{n/\alpha} T_{k/\beta} g \rangle = \begin{cases} 1 & \textit{if } (k,n) = (0,0) \\ 0 & \textit{otherwise} \end{cases}$

(c) $(\alpha\beta)^{-1} \langle M_{n/\alpha} T_{k/\beta} \gamma, M_{n'/\alpha} T_{k'/\beta} g \rangle = \begin{cases} 1 & \textit{if } (k,n) = (k',n') \\ 0 & \textit{otherwise} \end{cases}$

Remark. The preceding theorem completely characterizes all dual windows to a given window g when those windows are sufficiently regular. *Sufficiently regular* here and in the lemma preceding the theorem means that g and γ satisfy a mildly restrictive smoothness and decay condition. It is sufficient for the theorem for example that g and γ be in $W(\mathbf{R})$ or that the systems $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(\gamma,\alpha,\beta)$ have an upper frame bound.

B. The Ron-Shen Duality Principle.

This principle characterizes when a system $\mathcal{G}(g,\alpha,\beta)$ forms a frame by looking at the basis properties of the system $\mathcal{G}(g,1/\beta,1/\alpha)$. The proof of the theorem is somewhat involved and requires that we examine the detailed structure of the frame operator, as well as employ some standard results in operator theory.

Definition 1 *Given the Gabor system $\mathcal{G}(g,\alpha,\beta)$, define the operator $D\colon l^2(\mathbf{Z}^2)\to L^2(\mathbf{R})$ by*

$$Dc = \sum_{k,n} c_{k,n}\, T_{\alpha k} M_{\beta n} g.$$

In this case its adjoint $D^*\colon L^2(\mathbf{R})\to l^2(\mathbf{Z}^2)$ is given by $D^*f = \{\langle f, T_{\alpha k}M_{\beta n}g\rangle\}$. Note also that the frame operator $S_{g,g}$ associated to the Gabor system is given by $S_{g,g}=DD^*$. We will also write $D=D_{g,\alpha,\beta}$ when necessary for clarity.

Lemma 2 *The Gabor system $\mathcal{G}(g,\alpha,\beta)$ has an upper frame bound if and only if D (and also D^*) is bounded. Note that it is a standard result from operator theory that D is bounded if and only if D^* is bounded.*

The next lemma speaks to the detailed structure of the operators D and D^*.

Lemma 3 (a) $\|D^*_{g,\alpha,\beta}f\|^2_{l^2} = \dfrac{1}{\beta}\displaystyle\int_0^{1/\beta} \sum_{j,k} T_{k/\beta}f(x)\, T_{j/\beta}\overline{f(x)}\, G_{j,k}(x)\,dx$ *where*

$$G_{j,k}(x) = \sum_l \overline{g(x-k/\beta-\alpha l)}\, g(x-j/\beta-\alpha l)$$

and

(b) $\|D_{g,1/\beta,1/\alpha}c\|^2_2 = \displaystyle\int_0^{\alpha} \sum_{j,k} m_k(x)\,\overline{m_j(x)}\,\Gamma_{j,k}(x)\,dx$ *where*

$$\Gamma_{j,k}(x) = \sum_r g(x-k/\beta-\alpha r)\,\overline{g(x-j/\beta-\alpha r)} = \overline{G_{j,k}(x)}$$

and for each k, $m_k(x)=\sum_k c_{k,n}\, e^{2\pi i(n/\alpha)(x-k/\beta)}$.

Definition 2 *Given $G_{j,k}(x)$ and $\Gamma_{j,k}(x)$ as above we can define for each fixed x the operators $G(x)$ and $\Gamma(x)$ on the sequence space $l^2(\mathbf{Z})$ by*

$$(G(x)c)_j = \sum_k G_{j,k}(x)\, c_k \quad \text{and} \quad (\Gamma(x)c)_j = \sum_k \Gamma_{j,k}(x)\, c_k.$$

Note that these definitions simply treat $G_{j,k}$ and $\Gamma_{j,k}$ as bi-infinite matrices and that the operators defined above are the natural generalization of matrix-vector multiplication.

Considering now $G(x)$ and $\Gamma(x)$ as operators on l^2 for each x, let $\|G(x)\|_{op}$ and $\|\Gamma(x)\|_{op}$ denote their operator norms.

Lemma 4 *$D^*_{g,\alpha,\beta}$ is bounded if and only if $\sup_x \|G(x)\|_{op} < \infty$ and $D_{g,1/\beta,1/\alpha}$ is bounded if and only if $\sup_x \|\Gamma(x)\|_{op} < \infty$.*

We can now prove the following theorem which will be used in the proof of the Ron-Shen duality principle.

Theorem 3 *Given $g \in L^2(\mathbf{R})$, α, $\beta > 0$, the Gabor system $\mathcal{G}(g, \alpha, \beta)$ has an upper frame bound if and only if $\mathcal{G}(g, 1/\beta, 1/\alpha)$ has an upper frame bound.*

Theorem 4 *(Ron-Shen Duality Principle) Given $g \in L^2(\mathbf{R})$, α, $\beta > 0$, the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbf{R})$ if and only if the system $\mathcal{G}(g, 1/\beta, 1/\alpha)$ is a Riesz basis for its closed linear span.*

C. Corollaries regarding Density, Basis Properties, and Dual Windows.

Corollary 2 *If $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbf{R})$ then $\alpha\beta \leq 1$.*

Corollary 3 *$\mathcal{G}(g, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbf{R})$ if and only if it is a frame and $\alpha\beta = 1$.*

Corollary 4 *Let $\mathcal{G}(g, \alpha, \beta)$ be a frame for $L^2(\mathbf{R})$ and let $\gamma^o = S^{-1}_{g,g} g$ be the canonical dual window for this frame. Let $\mathcal{K}$ denote the closed linear span of the system $\mathcal{G}(g, 1/\beta, 1/\alpha)$. If γ is sufficiently regular, then the following hold.*

(a) *$\gamma^o \in \mathcal{K}$.*

(b) *γ is a dual window if and only if $\gamma \in \gamma^o + \mathcal{K}^\perp$.*

Corollary 5 *Let γ be a dual window for the Gabor frame $\mathcal{G}(g, \alpha, \beta)$. Then the following are equivalent*

(a) *$\gamma = \gamma^o$, the canonical dual window.*

(b) *$\|\gamma\|_2 < \|\gamma'\|_2$ for all dual windows $\gamma' \neq \gamma$.*

(c) $\left\| \dfrac{\gamma}{\|\gamma\|_2} - \dfrac{g}{\|g\|_2} \right\|_2 < \left\| \dfrac{\gamma'}{\|\gamma'\|_2} - \dfrac{g}{\|g\|_2} \right\|_2$ *for all dual windows $\gamma' \neq \gamma$.*

Remark. The final corollary states that the canonical dual window has minimum norm of all dual windows and that of all dual windows, it is closest to the original window in the sense that their normalizations are closest in norm.