
A. Definitions and basic properties.

Definition 1  Given \( a, b \in \mathbb{R} \) recall that the time-shift and frequency-shift operators \( T_a \) and \( M_b \) (respectively) on \( L^2(\mathbb{R}) \) are given by \( T_a f(x) = f(x - a) \) and \( M_b f(x) = e^{2\pi ibx} f(x) \). Note that \( (M_b f)(\gamma) = \hat{f}(\gamma - b) \).

Given a function \( g \in L^2(\mathbb{R}) \) and parameters \( \alpha, \beta > 0 \), the collection
\[
G(g, \alpha, \beta) = \{ T_{\alpha k} M_{\beta n} g : k, n \in \mathbb{Z} \}
\]
is called a Gabor system.

Remarks.
1. The function \( g \) is referred to as the window function for the Gabor system, and the numbers \( \alpha \) and \( \beta \) are the time and frequency-shift parameters respectively and are referred to as the lattice parameters collectively.
2. Historically, Gabor systems were introduced by D. Gabor who sought representations of functions in terms of time-frequency atoms with minimal support in the time-frequency plane. Consequently he proposed using \( g(x) = \varphi(x) = e^{-\pi x^2} \) which minimizes the uncertainty principle inequality and is concentrated in the unit square of the time frequency plane centered at the origin. Gabor wanted representations of functions \( f \) of the form
\[
f = \sum_k \sum_n c_{k,n} T_{\alpha k} M_{\beta n} g.
\]
3. The questions Gabor raised but did not answer included: (a) How do the coefficients \( \{c_{k,n}\} \) depend on the function \( f \)? (b) Are the coefficients in such a representation unique? (c) Do the coefficients depend in a stable way on the function \( f \)? It turns out that the most convenient setting in which to answer these questions uses the notion of a frame.

Definition 2  If the Gabor system \( G(g, \alpha, \beta) \) is a frame for \( L^2(\mathbb{R}) \) it is referred to as a Gabor frame. This means that there are constants \( 0 < A \leq B \) such that for all \( f \in L^2(\mathbb{R}) \),
\[
A \|f\|^2 \leq \sum_k \sum_n |\langle f, T_{\alpha k} M_{\beta n} g \rangle|^2 \leq B \|f\|^2.
\]

Associated to a Gabor frame is the Gabor frame operator \( S \) given by
\[
Sf = \sum_k \sum_n \langle f, T_{\alpha k} M_{\beta n} g \rangle T_{\alpha k} M_{\beta n} g.
\]

Lemma 1  The Gabor frame operator associated to a Gabor frame \( G(g, \alpha, \beta) \) commutes with the operators \( T_{\alpha k} \) and \( M_{\beta n} \) for all \( k, n \in \mathbb{Z} \).

Corollary 1  The dual frame associated to the Gabor frame \( G(g, \alpha, \beta) \) has the form \( G(S^{-1}g, \alpha, \beta) \), where \( S \) is the Gabor frame operator associated to the original frame. In other words, the dual frame to a Gabor frame is another Gabor frame. We call the function \( \gamma = S^{-1}g \) the dual window for the Gabor frame.
Lemma 2  (a) The Gabor system \( G(g, \alpha, \beta) \) is a frame for \( L^2(\mathbb{R}) \) if and only if the system \( G(\hat{g}, \beta, \alpha) \) is a frame for \( L^2(\mathbb{R}) \). Moreover each frame has the same frame bounds.

(b) The Gabor system \( G(g, \alpha, \beta) \) is a frame for \( L^2(\mathbb{R}) \) if and only if the system \( G(D_a g, \alpha', \beta') \) is a frame for \( L^2(\mathbb{R}) \) where \( D_a \) is the dilation operator \( D_a g(x) = a^{1/2} g(ax) \), \( \alpha' \beta' = \alpha \beta \) and \( a = \alpha / \alpha' = \beta' / \beta \). In other words, determining the existence of Gabor frames for given lattice parameters \( \alpha, \beta \) depends only on the product \( \alpha \beta \) and not on the value of the parameters themselves.

B. Existence of Gabor frames.

Example. For some \( a > 0 \), let \( g(x) = \alpha^{-1/2} 1_{[0, \alpha]} \). Then the Gabor system \( G(g, \alpha, \beta) \) is an orthonormal basis for \( L^2(\mathbb{R}) \) if \( \alpha \beta = 1 \).

Remark. Note that in this case if \( \alpha \beta > 1 \) then the system \( G(g, \alpha, \beta) \) is incomplete and that if \( \alpha \beta < 1 \) then the system is overcomplete, that is, a function in \( L^2(\mathbb{R}) \) has multiple representations in terms of the frame elements. More specifically it means that if a function is removed from the system, the remaining functions also form a frame for \( L^2(\mathbb{R}) \).

Theorem 1  Let \( g \in L^2(\mathbb{R}) \) and \( \alpha, \beta > 0 \) be such that:

(a) there exist constants \( A, B \) such that \( 0 < a \leq \sum_n |g(x - n\alpha)|^2 \leq b < \infty \), and

(a) \( g \) has compact support, with supp\((g) \subset I \subset \mathbb{R} \), where \( I \) is some interval of length \( 1/\beta \).

Then \( G(g, \alpha, \beta) \) is a Gabor frame for \( L^2(\mathbb{R}) \) with frame bounds \( \beta^{-1} a, \beta^{-1} b \).

Corollary 2  Suppose that in addition to the hypotheses of the Theorem, \( g \) satisfies \( 0 < \inf_{x \in I} |g(x)| \leq \sup_{x \in I} |g(x)| < \infty \). Then with \( \alpha \beta = 1 \), \( G(g, \alpha, \beta) \) is a Riesz basis for \( L^2(\mathbb{R}) \).

Remark. Note that if in the above theorem \( g \) is continuous on \( \mathbb{R} \), then in order for \( G(g, \alpha, \beta) \) to be a Gabor frame we must have \( \alpha \beta < 1 \), and that if \( \alpha \beta > 1 \) the Gabor system is incomplete in \( L^2(\mathbb{R}) \). If \( \alpha \beta = 1 \) then at best the Gabor system will be complete but will lack a lower frame bound.

Theorem 2  For \( G(g, \alpha, \beta) \) to be a Gabor frame for \( L^2(\mathbb{R}) \) it is necessary (but not sufficient) that there be constants \( a, b > 0 \) such that \( a \leq \sum_n |g(x - n\alpha)|^2 \leq b \).

C. Representation of the Gabor frame operator.

Definition 3  A function \( g \) is said to be in the space \( W(L^\infty, L^1) = W(\mathbb{R}) \) provided that

\[
\|g\|_W = \sum_{n \in \mathbb{Z}} \sup_{x \in [0,1]} |g(x - n)| < \infty.
\]
Note that we could also define the norm as

\[ \|g\|_{W,\alpha} = \sum_{n \in \mathbb{Z}} \sup_{x \in [0, \alpha]} |g(x - \alpha n)| \]

in the sense that the first quantity is finite if and only if the other one is. Indeed we have the inequalities

\[ \|g\|_{W,\alpha'} \leq 2 \|g\|_{W,\alpha} \quad \text{and} \quad \|g\|_{W,\alpha} \leq M \|g\|_{W,\alpha'} \]

whenever \( \alpha' \geq \alpha \) where \( M \) is the maximum number of intervals \([0, \alpha] + n\) that intersect any interval of the form \([0, \alpha'] + k\).

**Definition 4**

For \( g \in L^2(\mathbb{R}) \), \( \alpha, \beta > 0 \) and \( n \in \mathbb{Z} \), define the correlation function \( G_n(x) \) by

\[ G_n(x) = \sum_{k} g(x - n/\beta - \alpha k) g(x - \alpha k). \]

**Lemma 3**

If \( g \in W(\mathbb{R}) \) then

(a) \( \sum_{n} \|G_n\|_{\infty} \leq C(\alpha, \beta) \|g\|_{W}^2 \).

(b) For each fixed \( \alpha > 0 \), \( \lim_{\beta \to 0^+} \sum_{n \neq 0} \|G_n\|_{\infty} = 0 \).

**Theorem 3**

If \( g \in W(\mathbb{R}) \) and \( \alpha, \beta > 0 \), then the frame operator \( S \) associated to \( \mathcal{G}(g, \alpha, \beta) \) is given by

\[ Sf(x) = \frac{1}{\beta} \sum_{n} G_n(x) f(x - n/\beta). \]

In particular, \( S \) is bounded with \( \|S\| \leq \frac{1}{\beta} \sum_{n} \|G_n\|_{\infty} \).

**Theorem 4**

Let \( g \in W(\mathbb{R}) \), \( \alpha > 0 \) and suppose that there are constants \( a, b > 0 \) such that

\[ a \leq \sum_{k} |g(x - \alpha k)|^2 \leq b. \]

Then there is a \( \beta_0 > 0 \) such that \( \mathcal{G}(g, \alpha, \beta) \) is a Gabor frame for all \( 0 < \beta \leq \beta_0 \).

**Examples.**

1. Suppose that \( g \in L^2(\mathbb{R}) \) satisfies the hypotheses of Theorem 1. In this case the frame operator \( S \) is given simply by \( Sf(x) = \beta^{-1} f(x) G_0(x) \) so that \( S \) is an isomorphism of \( L^2(\mathbb{R}) \) if and only if \( G_0(x) \) is bounded above and away from zero. Also in this case \( S^{-1} \) is given by \( S^{-1} f(x) = \beta f(x) G_0^{-1}(x) \).

2. It is also clear from the previous theorem that for any \( \alpha > 0 \), \( \mathcal{G}(\varphi, \alpha, \beta) \), where \( \varphi(x) = e^{-\pi x^2} \) is a Gabor frame for all \( \beta > 0 \) small enough. An interesting question is: How large can \( \beta \) be in order for the Gabor system to be a frame?