Gabor Systems: Existence and Basic Properties.

A. Definitions and basic properties.

Definition 1 *Given a, $b \in \mathbf{R}$ recall that the* time-shift *and* frequency-shift *operators T_a and M_b (respectively) on $L^2(\mathbf{R})$ are given by $T_a f(x) = f(x-a)$ and $M_b f(x) = e^{2\pi i b x} f(x)$. Note that $(M_b f)^\wedge(\gamma) = \widehat{f}(\gamma - b)$.*

Given a function $g \in L^2(\mathbf{R})$ and parameters α, $\beta > 0$, the collection

$$\mathcal{G}(g, \alpha, \beta) = \{T_{\alpha k} M_{\beta n} g \colon k,\, n \in \mathbf{Z}\}$$

is called a Gabor system*.*

Remarks.

1. The function g is referred to as the *window function* for the Gabor system, and the numbers α and β are the *time* and *frequency-shift parameters* respectively and are referred to as the *lattice parameters* collectively.
2. Historically, Gabor systems were introduced by D. Gabor who sought representations of functions in terms of time-frequency atoms with minimal support in the time-frequency plane. Consequently he proposed using $g(x) = \varphi(x) = e^{-\pi x^2}$ which minimizes the uncertainty principle inequality and is concentrated in the unit square of the time frequency plane centered at the origin. Gabor wanted representations of functions f of the form $f = \sum_k \sum_n c_{k,n} T_k M_n g$.
3. The questions Gabor raised but did not answer included: (a) How do the coefficients $\{c_{k,n}\}$ depend on the function f? (b) Are the coefficients in such a representation unique? (c) Do the coefficients depend in a stable way on the function f? It turns out that the most convenient setting in which to answer these questions uses the notion of a frame.

Definition 2 *If the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbf{R})$ it is referred to as a* Gabor frame*. This means that there are constants $0 < A \leq B$ such that for all $f \in L^2(\mathbf{R})$,*

$$A \, \|f\|^2 \leq \sum_k \sum_n |\langle f, T_{\alpha k} M_{\beta n} g \rangle|^2 \leq B \, \|f\|^2.$$

Associated to a Gabor frame is the Gabor frame operator *S given by*

$$Sf = \sum_k \sum_n \langle f, T_{\alpha k} M_{\beta n} g \rangle \, T_{\alpha k} M_{\beta n} g.$$

Lemma 1 *The Gabor frame operator associated to a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ commutes with the operators $T_{\alpha k}$ and $M_{\beta n}$ for all k, $n \in \mathbf{Z}$.*

Corollary 1 *The dual frame associated to the Gabor frame $\mathcal{G}(g, \alpha, \beta)$ has the form $\mathcal{G}(S^{-1} g, \alpha, \beta)$, where S is the Gabor frame operator associated to the original frame. In other words, the dual frame to a Gabor frame is another Gabor frame. We call the function $\gamma = S^{-1} g$ the* dual window *for the Gabor frame.*

Lemma 2 (a) *The Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbf{R})$ if and only if the system $\mathcal{G}(\widehat{g}, \beta, \alpha)$ is a frame for $L^2(\mathbf{R})$. Moreover each frame has the same frame bounds.*

(b) *The Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbf{R})$ if and only if the system $\mathcal{G}(D_a g, \alpha', \beta')$ is a frame for $L^2(\mathbf{R})$ where D_a is the dilation operator $D_a g(x) = a^{1/2} g(ax)$, $\alpha'\beta' = \alpha\beta$ and $a = \alpha/\alpha' = \beta'/\beta$. In other words, determining the existence of Gabor frames for given lattice parameters α, β depends only on the product $\alpha\beta$ and not on the value of the parameters themselves.*

B. Existence of Gabor frames.

Example. For some $a > 0$, let $g(x) = \alpha^{-1/2} \mathbf{1}_{[0,\alpha]}$. Then the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is an orthonormal basis for $L^2(\mathbf{R})$ if $\alpha\beta = 1$.

Remark. Note that in this case if $\alpha\beta > 1$ then the system $\mathcal{G}(g, \alpha, \beta)$ is incomplete and that if $\alpha\beta < 1$ then the system is overcomplete, that is, a function in $L^2(\mathbf{R})$ has multiple representations in terms of the frame elements. More specifically it means that if a function is removed from the system, the remaining functions also form a frame for $L^2(\mathbf{R})$.

Theorem 1 *Let $g \in L^2(\mathbf{R})$ and $\alpha, \beta > 0$ be such that:*

(a) *there exist constants A, B such that $0 < a \leq \sum_n |g(x - n\alpha)|^2 \leq b < \infty$, and*

(a) *g has compact support, with $\operatorname{supp}(g) \subset I \subset \mathbf{R}$, where I is some interval of length $1/\beta$. Then $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbf{R})$ with frame bounds $\beta^{-1}a$, $\beta^{-1}b$.*

Corollary 2 *Suppose that in addition to the hypotheses of the Theorem, g satisfies $0 < \inf_{x \in I} |g(x)| \leq \sup_{x \in I} |g(x)| < \infty$. Then with $\alpha\beta = 1$, $\mathcal{G}(g, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbf{R})$.*

Remark. Note that if in the above theorem g is continuous on $\mathbf{R}$, then in order for $\mathcal{G}(g, \alpha, \beta)$ to be a Gabor frame we must have $\alpha\beta < 1$, and that if $\alpha\beta > 1$ the Gabor system is incomplete in $L^2(\mathbf{R})$. If $\alpha\beta = 1$ then at best the Gabor system will be complete but will lack a lower frame bound.

Theorem 2 *For $\mathcal{G}(g, \alpha, \beta)$ to be a Gabor frame for $L^2(\mathbf{R})$ it is necessary (but not sufficient) that there be constants a, $b > 0$ such that $a \leq \sum_n |g(x - n\alpha)|^2 \leq b$.*

C. Representation of the Gabor frame operator.

Definition 3 *A function g is said to be in the space $W(L^\infty, L^1) = W(\mathbf{R})$ provided that*

$$\|g\|_W = \sum_{n \in \mathbf{Z}} \sup_{x \in [0,1]} |g(x - n)| < \infty.$$

Note that we could also define the norm as

$$\|g\|_{W,\alpha} = \sum_{n\in\mathbf{Z}} \sup_{x\in[0,\alpha]} |g(x-\alpha n)|$$

in the sense that the first quantity is finite if and only if the other one is. Indeed we have the inequalities

$$\|g\|_{W,\alpha'} \leq 2\,\|g\|_{W,\alpha} \quad \text{and} \quad \|g\|_{W,\alpha} \leq M\,\|g\|_{W,\alpha'}$$

whenever $\alpha' \geq \alpha$ where M is the maximum number of intervals $[0,\alpha]+n$ that intersect any interval of the form $[0,\alpha']+k$.

Definition 4 *For $g \in L^2(\mathbf{R})$, α, $\beta > 0$ and $n \in \mathbf{Z}$, define the* correlation function *$G_n(x)$ by*

$$G_n(x) = \sum_k \overline{g(x-n/\beta-\alpha k)}\, g(x-\alpha k).$$

Lemma 3 *If $g \in W(\mathbf{R})$ then*

(a) $\sum_n \|G_n\|_\infty \leq C(\alpha,\beta)\,\|g\|_W^2$,

(b) *for each fixed $\alpha > 0$, $\lim_{\beta\to 0^+} \sum_{n\neq 0} \|G_n\|_\infty = 0$.*

Theorem 3 *If $g \in W(\mathbf{R})$ and α, $\beta > 0$, then the frame operator S associated to $\mathcal{G}(g,\alpha,\beta)$ is given by*

$$Sf(x) = \frac{1}{\beta}\sum_n G_n(x)\, f(x-n/\beta).$$

In particular, S is bounded with $\|S\| \leq \frac{1}{\beta}\sum_n \|G_n\|_\infty$.

Theorem 4 *Let $g \in W(\mathbf{R})$, $\alpha > 0$ and suppose that there are constants a, $b > 0$ such that*

$$a \leq \sum_k |g(x-\alpha k)|^2 \leq b.$$

Then there is a $\beta_0 > 0$ such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for all $0 < \beta \leq \beta_0$.

Examples.

1. Suppose that $g \in L^2(\mathbf{R})$ satisfies the hypotheses of Theorem 1. In this case the frame operator S is given simply by $Sf(x) = \beta^{-1} f(x)\, G_0(x)$ so that S is an isomorphism of $L^2(\mathbf{R})$ if and only if $G_0(x)$ is bounded above and away from zero. Also in this case S^{-1} is given by $S^{-1}f(x) = \beta f(x)\, G_0^{-1}(x)$.
2. It is also clear from the previous theorem that for any $\alpha > 0$, $\mathcal{G}(\varphi,\alpha,\beta)$, where $\varphi(x) = e^{-\pi x^2}$ is a Gabor frame for all $\beta > 0$ small enough. An interesting question is: How large can β be in order for the Gabor system to be a frame?