# QMF Conditions and the DWT.

**A. Quick tutorial on Fourier Analysis of sequences.**

We have seen that if $f(x)$ has period 1 and is in $L^2[0,1]$, then we can expand $f$ in its Fourier series as

$$f(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x}$$

where $\widehat{f}(n) = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$ are the Fourier coefficients of $f$. Since the collection of exponentials is an orthonormal basis for $L^2[0,1]$ we have Plancherel's formula

$$\|f\|_2^2 = \int_0^1 |f(x)|^2\, dx = \sum_n |\widehat{f}(n)|^2.$$

We can change our perspective and define a Fourier analysis for sequences as follows.

**Definition 0.1** *Given a sequence* $\{c(n)\}_{n\in\mathbf{Z}}$ *in* $\ell^2$ *we define its* frequency representation *(or its* Fourier transform*),* $C(\gamma)$ *(or sometimes* $\widehat{c}(\gamma)$*), a period 1 function in* $L^2[0,1]$*, by*

$$C(\gamma) = \sum_n c(n)\, e^{-2\pi i n\gamma}.$$

We have as with Fourier series and Fourier transforms the *inversion formula*

$$c(n) = \int_0^1 C(\gamma)\, e^{2\pi i n\gamma}\, d\gamma,$$

and *Plancherel's formula*

$$\sum_n |c(n)|^2 = \int_0^1 |C(\gamma)|^2\, d\gamma.$$

We also have the notions of translation and convolution for sequences.

**Theorem 0.1** *Let* $c = \{c(n)\}_{n\in\mathbf{Z}} \in \ell^2$*. The* shift operator, $\tau$ *is defined on* $\ell^2$ *by* $(\tau_m c)(n) = c(n-m)$*. The Fourier transform of* $\tau_m c$ *is* $e^{-2\pi i n\gamma}\, C(\gamma)$*.*

**Theorem 0.2** *Given two sequences* $c = \{c(n)\}_{n\in\mathbf{Z}}$ *and* $d = \{d(n)\}_{n\in\mathbf{Z}}$ *both in* $\ell^1$*, their convolution* $c * d$ *is the sequence*

$$(c*d)(n) = \sum_k c(k)\, d(n-k) = \sum_k d(k)\, c(n-k).$$

*This sequence is also in* $\ell^1$ *and its Fourier transform is given by* $C(\gamma)D(\gamma)$*.*

**B. Motivation: Wavelet Expansions of Discrete Data.**

We start with the assumption that we are given a wavelet $\psi(x)$ that comes from a MRA with scaling function $\varphi(x)$ with scaling filter $\{h(n)\}$. Suppose we are given a sequence of data $c_0 = \{c_0(n)\}_{n\in\mathbf{Z}}$. We will make several observations:

1. That we assume $c_0$ is an infinite sequence makes it no less interesting for practical applications. Anything we can say for an infinite sequence we can also say of any finite sequence regardless of its length.

2. In most digital signal processing (DSP) applications, a sequence of discrete data is assumed to be the samples of some underlying bandlimited function or signal. Hence in principle the discrete data completely determines the underlying function or signal. The assumption that the signal is bandlimited is really only for convenience and comes from the assumption that the discrete data consists of samples.

3. A more natural assumption for our purposes is that the data represent the level zero scaling coefficients of some underlying function $f$. In other words, we assume that the actual signal we are measuring is given by $f = \sum_n c_0(n)\,\varphi_{0,n}$, so that $f \in V_0$. In this case once again the data completely determine the underlying function but a different assumption of convenience is made.

4. Under the above assumption, it is useless to try to compute the projections $P_j f$ and $Q_j f$ for $j > 0$ since $P_j f = f$ and $Q_j f = 0$ for all such $j$. Therefore the only wavelet decomposition we can do is to find $P_j f$ and $Q_j f$ for $j \leq 0$. Below we will see how to do that.

**Lemma 0.1** *Since $\varphi_{0,0} = \sum_n h(n)\,\varphi_{1,n}$, it follows that*

(a) $\varphi_{0,k} = \sum\limits_n h(n-2k)\,\varphi_{1,n}$,

(b) $\varphi_{j,k} = \sum\limits_n h(n-2k)\,\varphi_{j+1,n}$,

(c) $\psi_{j,k} = \sum\limits_n g(n-2k)\,\varphi_{j+1,n}$.

**Lemma 0.2** *If $c_j(k) = \langle f, \varphi_{-j,k}\rangle$ and $d_j(k) = \langle f, \psi_{-j,k}\rangle$, then*

(a) $c_{j+1}(k) = \sum\limits_n c_j(n)\,\overline{h(n-2k)}$,

(b) $d_{j+1}(k) = \sum\limits_n c_j(n)\,\overline{g(n-2k)}$,

(c) $c_j(k) = \sum\limits_n c_{j+1}(n)\,h(k-2n) + \sum\limits_n d_{j+1}(n)\,g(k-2n)$.

**Remark.** Note that the wavelet and scaling coefficients of $f$ at all negative scales can be computed from the initial scale-zero scaling coefficients and the scaling and wavelet filters, and that this process is completely reversible. Our goal will be to back out the properties of the scaling and wavelet filters that allow this to hold. This yields a theory of wavelets completely in the discrete domain.

**Theorem 0.3** *Let $\{V_j\}$ be an MRA with scaling filter $h(k)$ and wavelet filter $g(k)$. Then*

(a) $\sum_n h(n) = \sqrt{2} \quad (\Longleftrightarrow \int \varphi(x)\,dx \neq 0)$

(b) $\sum_n g(n) = 0 \quad (\Longleftrightarrow \int \psi(x)\,dx = 0)$

(c) $\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n)$ $(\Longleftrightarrow \langle \varphi_{0,0}, \varphi_{0,n}\rangle = \langle \psi_{0,0}, \psi_{0,n}\rangle = \delta(n))$

(d) $\sum_k g(k)\,\overline{h(k-2n)} = 0$ *for all* $n \in \mathbf{Z}$ $(\Longleftrightarrow \langle \varphi_{0,0}, \psi_{0,n}\rangle = 0$, *all* $n)$

(e) $\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(n-m)$ $(\Longleftrightarrow P_{j+1} = P_j + Q_j)$.

## C. Approximation and Detail Operators.

**Definition 0.2** *Given a filter $h(k)$, let $g(k) = (-1)^k\,\overline{h(1-k)}$. The* approximation operator $H$ *and* detail operator $G$ *corresponding to $h(k)$ are given by*

(a) $(Hc)(k) = \sum_n c(n)\,\overline{h(n-2k)}$,

(b) $(Gc)(k) = \sum_n c(n)\,\overline{g(n-2k)}$.

*Define the* approximation adjoint $H^*$ *and* detail adjoint $G^*$ *by*

(c) $(H^*c)(k) = \sum_n c(n)\,h(k-2n)$,

(d) $(G^*c)(k) = \sum_n c(n)\,g(k-2n)$.

**Theorem 0.4** *Given $h(k)$, $g(k)$ as above,*

(a) $\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n) \Longleftrightarrow HH^* = GG^* = I$,

(b) $\sum_k g(k)\,\overline{h(k-2n)} = 0 \Longleftrightarrow HG^* = GH^* = 0$,

(c) $\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(m-n) \Longleftrightarrow H^*H + G^*G = I$.

The operators $H$ and $G$ are transformations on the Hilbert space $\ell^2$ of square-summable sequences, and we can talk about how these operators behave in the *transform domain*, that is, we can look at the Fourier transforms of $Hc$ and $Gc$ for $c \in \ell^2$.

**Definition 0.3** *Let $c(n)$ be in $\ell^1$.*

(a) *The* downsampling operator $\downarrow$ *is defined by* $(\downarrow c)(n) = c(2n)$.

(b) *The* upsampling operator $\uparrow$ *is defined by*

$$(\uparrow c)(n) = \begin{cases} c(n/2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

**Lemma 0.3** *Given $c(n)$ in $\ell^1$,*

(a) $(\downarrow c)^\wedge(\gamma) = \frac{1}{2}\Big(\widehat{c}\left(\frac{\gamma}{2}\right) + \widehat{c}\left(\frac{\gamma+1}{2}\right)\Big)$.

(b) $(\uparrow c)^\wedge(\gamma) = \widehat{c}(2\gamma)$.

**Lemma 0.4** (a) *Defining $\underline{h}(n) = \overline{h(-n)}$ and $\underline{g}(n) = \overline{g(-n)}$ (also known as the* involution *of $h$ and $g$), then $(Hc)(n) = \downarrow(c * \underline{h})(n)$ and $(Gc)(n) = \downarrow(c * \underline{g})(n)$.*

(b) *Also $(H^*c)(n) = (\uparrow c) * h(n)$ and $(G^*c)(n) = (\uparrow c) * g(n)$.*

**Lemma 0.5** *Given $h(k)$, $g(k) = (-1)^k\, \overline{h(1-k)}$, $m_0(\gamma) = 2^{-1/2} \sum_k h(k)\, e^{-2\pi i k \gamma}$, and $m_1(\gamma) = 2^{-1/2} \sum_k g(k)\, e^{-2\pi i k \gamma}$. Then for any $c(n)$,*

(a) $(Hc)^\wedge(\gamma) = \dfrac{1}{\sqrt{2}} \Big(\widehat{c}(\gamma/2)\, \overline{m_0(\gamma/2)} + \widehat{c}(\gamma/2 + 1/2)\, \overline{m_0(\gamma/2 + 1/2)}\Big)$,

(b) $(Gc)^\wedge(\gamma) = \dfrac{1}{\sqrt{2}} \Big(\widehat{c}(\gamma/2)\, \overline{m_1(\gamma/2)} + \widehat{c}(\gamma/2 + 1/2)\, \overline{m_1(\gamma/2 + 1/2)}\Big)$,

(c) $(H^*c)^\wedge(\gamma) = \sqrt{2}\widehat{c}(2\gamma) m_0(\gamma)$, *and* $(G^*c)^\wedge(\gamma) = \sqrt{2}\widehat{c}(2\gamma) m_1(\gamma)$.

**Lemma 0.6** *Given $h(k)$, $g(k)$ as usual. Then $m_0(\gamma)\, \overline{m_0(\gamma + 1/2)} + m_1(\gamma)\, \overline{m_1(\gamma + 1/2)} = 0$ which is equivalent to $HG^* = GH^* = 0$.*

**Theorem 0.5** *Given $h(k)$, $g(k)$, $m_0(\gamma)$, $m_1(\gamma)$, and the operators $H$, $G$, $H^*$, and $G^*$ as above, the following are equivalent.*

(a) $|m_0(\gamma)|^2 + |m_0(\gamma + 1/2)|^2 \equiv 1$.

(b) $H^*H + G^*G = I$.

(c) $HH^* = GG^* = I$.

## D. The QMF Conditions.

**Definition 0.4** *Given $h(k)$, $m_0(\gamma)$ as before, we say $h(k)$ is a QMF (quadrature mirror filter) if*

(a) $m_0(0) = 1$ *and*

(b) $|m_0(\gamma/2)|^2 + |m_0(\gamma/2 + 1/2)|^2 \equiv 1.$

**Theorem 0.6** *Suppose that $h(k)$ is a QMF and define $g(k)$ as before. Then:*

(a) $\sum_n h(n) = \sqrt{2},$

(b) $\sum_n g(n) = 0,$

(c) $\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n).$

(d) $\sum_k g(k)\,\overline{h(k-2n)} = 0$ *for all* $n \in \mathbf{Z}$.

(e) $\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(n-m).$

**E. The Discrete Wavelet Transform (DWT).**

**1. For infinite signals:** Let $h(k)$ be a QMF, $g(k)$ the dual filter, and let $H$, $G$, $H^*$, and $G^*$ be as above. Fix $J \in \mathbf{N}$. The DWT of a signal $c_0(n)$, is the collection of sequences

$$\{d_j(k)\colon 1 \le j \le J;\ k \in \mathbf{Z}\} \cup \{c_J(k)\colon k \in \mathbf{Z}\},$$

where $c_{j+1}(n) = (Hc_j)(n)$, and $d_{j+1}(n) = (Gc_j)(n)$. The inverse transform is $c_j(n) = (H^*c_{j+1})(n) + (G^*d_{j+1})(n)$. If $J = \infty$, then the DWT of $c_0$ is the collection of sequences

$$\{d_j(k)\colon j \in \mathbf{N};\ k \in \mathbf{Z}\}.$$

**2. For finite, zero-padded signals:** Suppose that $c_0(n)$ has length $2^N$, and that $h(n)$ and $g(n)$ have length $L > 2$, with $L$ even. Then

(a) $c_1 = Hc_0$ and $d_1 = Gc_0$ each have length $(2^N + L - 2)/2$,

(b) $c_j$ and $d_j$ would have length at least $2^{N-j} + (1 - 2^{-j})(L - 2)$.

(c) The total length of the DWT for $c_0$ would be at least $(2^N\,2^{-J} + (1 - 2^{-J})(L-2)) + \sum_{j=1}^{J}(2^N\,2^{-j} + (1 - 2^{-j})(L-2)) = 2^N + J(L-2)$, where $J \in \mathbf{N}$ indicates the depth chosen for the DWT.

**3. For periodic signals:**

**Lemma 0.7** *Let $c(n)$ have period $2^N$, $h(k)$ a QMF, Then $(Hc)(n)$ and $(Gc)(n)$ have period $2^{N-1}$, and $(H^*c)(n)$ and $(G^*c)(n)$ have period $2^{N+1}$.*

**MATLAB illustration for zero-padded signals.**

```
>> x=[0 1 2 3 4 5 6 7 8 7 6 5 4 3 2 1];
>> dwtmode('zpd')

****************************************
**  DWT Extension Mode: Zero Padding  **
****************************************

>> [h g h1 g1]=wfilters('db2');
>> h
h =
   -0.1294    0.2241    0.8365    0.4830
>> [c1 d1]=dwt(x,'db2')
c1 =
   -0.1294    0.8966    3.7250    6.5534    9.6407
   10.4171    7.5887    4.7603    1.8024
d1 =
   -0.4830   -0.0000   -0.0000   -0.0000    0.9659
   0.0000    0.0000    0.0000   -0.4830
>> length(x)
ans =
    16
>> length([c1 d1])
ans =
    18
>> [C L]=wavedec(x,4,'db2');
>> length(C)
ans =
    25
```

**MATLAB illustration for periodic signals:**

```
>> dwtmode('per')
*****************************************
**  DWT Extension Mode: Periodization  **
*****************************************
>> [c1 d1]=dwt(x,'db2')
c1 =
    0.4483    2.3108    5.1392    7.9676
    10.8654    9.0029    6.1745    3.3461
d1 =
   -0.2588   -0.0000   -0.0000   -0.0000
   0.2588    0.0000    0.0000    0.0000
>> length(x)
ans =
```

```
    16
>> length([c1 d1])
ans =
    16
>> [C L]=wavedec(x,4,'db2');
>> length(C)
ans =
    16
```