

Multiresolution Analysis.

A. Definition and Examples.

Definition 0.1 A multiresolution analysis on $\mathbf{R}$ is a sequence of subspaces $\{V_j\}_{j \in \mathbf{Z}} \subseteq L^2(\mathbf{R})$ satisfying:

- (a) For all $j \in \mathbf{Z}$, $V_j \subseteq V_{j+1}$.
- (b) $\overline{\text{span}}\{V_j\}_{j \in \mathbf{Z}} = L^2(\mathbf{R})$. That is, the set $\cup_{j \in \mathbf{Z}} V_j$ is dense in $L^2(\mathbf{R})$.
- (c) $\cap_{j \in \mathbf{Z}} V_j = \{0\}$.
- (d) A function $f(x) \in V_0$ if and only if $D_{2^j} f(x) \in V_j$.
- (e) There exists a function $\varphi(x)$, L^2 on $\mathbf{R}$, called the scaling function such that the collection $\{T_n \varphi(x)\}$ is an orthonormal basis for V_0 .

Remarks. (a) An MRA is completely determined by the scaling function $\varphi(x)$ in the following way. Given φ with the property that $\{T_n \varphi(x)\}$ is an orthonormal system, define the subspace V_0 by $V_0 = \overline{\text{span}}\{T_n \varphi(x)\}$, and the subspaces V_j by $V_j = D_{2^j} V_0$, that is, $f \in V_j$ if and only if $D_{2^{-j}} f \in V_0$. Then verify that (a)–(e) hold for this sequence of subspaces.

(b) For example, if we let $\varphi(x) = \chi_{[0,1]}(x)$, then the MRA so generated is called the *Haar MRA* and leads to the construction of the Haar wavelet.

(c) If we let $\varphi(x)$ be defined by $\hat{\varphi}(\gamma) = \chi_{[-1/2, 1/2]}(\gamma)$, then the MRA so generated is called the *bandlimited MRA* and leads to the construction of the Bandlimited wavelet.

(d) If we let $\varphi(x)$ be defined by

$$\hat{\varphi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \geq 4/3 \\ 1 & \text{if } |\gamma| \leq 1/3 \\ c(\gamma - 1/2) & \text{if } x \in (1/3, 2/3) \\ s(\gamma + 1/2) & \text{if } x \in (-2/3, -1/3) \end{cases}$$

where $s(\gamma)$ and $c(\gamma)$ are as defined in Lecture 3, then the MRA so generated is called the *Meyer MRA* and leads to the construction of the Meyer wavelet.

Lemma 0.1 Given $\varphi \in L^2(\mathbf{R})$, the system $\{T_n \varphi(x)\}$ is an orthonormal system if and only if

$$\sum_{n \in \mathbf{Z}} |\hat{\varphi}(\gamma + n)|^2 \equiv 1.$$

Corollary 0.1 Suppose that the system $\{T_n \varphi(x)\}$ is an orthonormal system. Then a function $f \in \overline{\text{span}}\{T_n \varphi(x)\}$ if and only if there is a sequence $\{c_n\} \in l^2$ such that $f = \sum_n c_n T_n \varphi$ or equivalently if there is a period-1 function $C \in L^2[0, 1]$ such that $\hat{f}(\gamma) = C(\gamma) \hat{\varphi}(\gamma)$. (Exercise 1. Verify this.)

Lemma 0.2 $\{D_{2^j}T_k\varphi\}_{k \in \mathbf{Z}}$ is an orthonormal basis for V_j . (**Exercise 2. Verify this.**)

Lemma 0.3 There exists $\{h(k)\} \in \ell^2$ such that $\varphi(x) = \sum_k h(k) 2^{1/2} \varphi(2x - k)$. This equation is referred to as the two-scale dilation equation and the sequence $\{h(k)\}$ is referred to as the scaling sequence or scaling filter. We may write $\hat{\varphi}(\gamma) = m_0(\gamma/2) \hat{\varphi}(\gamma/2)$, where $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k) e^{-2\pi i k \gamma}$ is called the auxiliary function. (**Exercise 3. Verify the last statement in this Lemma.**)

Lemma 0.4 If $\{T_n\varphi(x)\}$ is an orthonormal system and if $\varphi(x)$ satisfies the two-scale dilation equation with scaling filter $\{h(k)\}$. Then the auxiliary function $m_0(\gamma)$ satisfies

$$|m_0(\gamma)|^2 + |m_0(\gamma + 1/2)|^2 \equiv 1.$$

Remark. Our goal in this lecture is to prove the following theorem, which will be restated in more detail below: If $\{V_j\}$ is an MRA, then there exists a function $\psi \in L^2(\mathbf{R})$ such that $\{\psi_{j,k}\}$ is an orthonormal wavelet basis for $L^2(\mathbf{R})$.

B. Projectors and Subspaces.

Recall that if H is a Hilbert space and M a closed subspace of H , then given $x \in H$, we can decompose x uniquely as $x = x_M + x_{M^\perp}$ where $x_M \in M$ and $x_{M^\perp} \in M^\perp$. The operator P_M that maps x to x_M is called the *orthogonal projector onto M* . The basic result here is the following.

Theorem 0.1 An operator P on a Hilbert space H is the orthogonal projector onto $\text{Ran}(P)$ if and only if $P^2 = P$ and P is self-adjoint, that is, $\langle Px, y \rangle = \langle x, Py \rangle$ for all $x, y \in H$. Also, if P is an orthogonal projector then for all $x \in H$, $\|Px\| \leq \|x\|$.

Definition 0.2 For each $j \in \mathbf{Z}$, define the approximation operator P_j to be the orthogonal projector onto V_j and the detail operator Q_j by $Q_j = P_{j+1} - P_j$. Let W_j be the orthogonal complement of V_j in V_{j+1} . In other words, W_j consists of all $f \in V_{j+1}$ such that $\langle f, g \rangle = 0$ for all $g \in V_j$.

Lemma 0.5 $P_j f = \sum_k \langle f, \varphi_{j,k} \rangle \varphi_{j,k}$.

Lemma 0.6 For all $f \in L^2(\mathbf{R})$, $P_j f \rightarrow f$ and $P_{-j} f \rightarrow 0$ as $j \rightarrow \infty$.

Lemma 0.7 Q_j is the orthogonal projector onto W_j .

C. Recipe for constructing wavelet bases.

Theorem 0.2 Let $\{V_j\}$ be an MRA with scaling function $\varphi(x)$ and scaling filter $h(k)$. Define the wavelet filter $g(k)$ by $g(k) = (-1)^k \overline{h(1-k)}$ and the wavelet $\psi(x)$ by $\psi(x) = \sum_k g(k) 2^{1/2} \varphi(2x - k)$. Then $\{\psi_{j,k}(x)\}_{j,k \in \mathbf{Z}}$ is a wavelet orthonormal basis on $\mathbf{R}$.

Alternatively, given any $J \in \mathbf{Z}$,

$$\{\varphi_{J,k}(x)\}_{k \in \mathbf{Z}} \cup \{\psi_{j,k}(x)\}_{j,k \in \mathbf{Z}}$$

is an orthonormal basis on $\mathbf{R}$.

Remarks. (a) Taking the Fourier transform gives that $\widehat{\psi}(\gamma) = m_1(\gamma/2) \widehat{\varphi}(\gamma/2)$, where $m_1(\gamma) = e^{-2\pi i(\gamma+1/2)} \overline{m_0(\gamma+1/2)}$. (**Exercise 4. Verify this statement.**)

(b) The alternate form for the wavelet basis is based on the observation that in a traditional wavelet basis, the “low frequency” components of a function f , that is, that portion of $\widehat{f}(\gamma)$ for γ near zero, are the coefficients $\{\langle f, \psi_{j,k} \rangle\}_{j < J, k \in \mathbf{Z}}$ where J is any fixed integer. These components can be equivalently and more conveniently expressed in the coefficients $\{\langle f, \varphi_{J,k} \rangle\}_{k \in \mathbf{Z}}$.

The proof of the theorem is accomplished by proving the following lemmas.

Lemma 0.8 Suppose that $\psi \in L^2$ has the property that $\{T_n \psi\}_{n \in \mathbf{Z}}$ is an orthonormal basis for W_0 . Then $\{\psi_{j,k}\}_{j,k \in \mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$.

Claim 1: $\{D_{2^j} T_n \psi\}_{n \in \mathbf{Z}}$ is an orthonormal basis for W_j . (**Exercise 5. Verify this claim.**)

Claim 2: $\text{span}\{D_{2^j} T_n \psi\}$ is dense in $L^2(\mathbf{R})$.

Lemma 0.9 Suppose that $\psi(x)$ is given by the recipe in the Theorem. Then $\{T_n \psi\}_{n \in \mathbf{Z}}$ is an orthonormal basis for W_0 .

Claim 1: With ψ so defined, the collection $\{T_n \psi\}_{n \in \mathbf{Z}}$ is an orthonormal set. (Exercise 6. Verify this. Hint: Use Lemma 0.1.)

Claim 2: $W_0 = \overline{\text{span}}\{T_n \psi\}_{n \in \mathbf{Z}}$.

Examples. (a) *The Haar wavelet.* In this case, we can compute the scaling and wavelet filters directly.

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) + \frac{1}{\sqrt{2}}\varphi_{1,1}(x).$$

Therefore,

$$h(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, 1, \\ 0 & \text{if } n \neq 0, 1, \end{cases}$$

Therefore,

$$g(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, \\ -\frac{1}{\sqrt{2}} & \text{if } n = 1, \\ 0 & \text{if } n \neq 0, 1. \end{cases}$$

and

$$\psi(x) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) - \frac{1}{\sqrt{2}}\varphi_{1,1}(x) = \varphi(2x) - \varphi(2x-1) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x).$$

(b) *The Bandlimited wavelet.* Here it is more convenient to work on the transform side. Recall that $\widehat{\varphi}(\gamma) = \chi_{[-1/2,1/2)}(\gamma)$. Since $\widehat{\varphi}(\gamma/2) = \chi_{[-1,1)}(\gamma)$, it follows that

$$\widehat{\varphi}(\gamma) = m_0(\gamma/2) \widehat{\varphi}(\gamma/2),$$

where $m_0(\gamma)$ is the period 1 extension of $\chi_{[-1/4,1/4)}(\gamma)$.

Thus, $m_1(\gamma)$ is the period 1 extension of the function

$$e^{-2\pi i(\gamma+1/2)} \left(\chi_{[-1/2, -1/4)}(\gamma) + \chi_{[1/4, 1/2)}(\gamma) \right)$$

so that

$$\widehat{\psi}(\gamma) = m_1(\gamma/2) \widehat{\varphi}(\gamma/2) = -e^{-\pi i \gamma} \left(\chi_{[-1, -1/2)}(\gamma) + \chi_{[1/2, 1)}(\gamma) \right).$$

By taking the inverse Fourier transform,

$$\psi(x) = \frac{\sin(2\pi x) - \cos(\pi x)}{\pi(x - 1/2)} = \frac{\sin \pi(x - 1/2)}{\pi(x - 1/2)} (1 - 2 \sin \pi x).$$

(c) *The Meyer wavelet.* Recall that

$$\widehat{\varphi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \geq 2/3, \\ 1 & \text{if } |\gamma| \leq 1/3, \\ s(\gamma + 1/2) & \text{if } \gamma \in (1/3, 2/3), \\ c(\gamma - 1/2) & \text{if } \gamma \in (-2/3, -1/3), \end{cases}$$

Therefore, $\widehat{\varphi}(\gamma) = m_0(\gamma/2) \widehat{\varphi}(\gamma/2)$, where $m_0(\gamma)$ is the period 1 extension of the function

$$\widehat{\varphi}(2\gamma) \chi_{[-1/2, 1/2]}(\gamma).$$

$\psi(x)$ is defined by

$$\widehat{\psi}(\gamma) = -e^{-\pi i \gamma} \overline{m_0(\gamma/2 + 1/2)} \widehat{\varphi}(\gamma/2)$$

and

$$\widehat{\psi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \leq 1/3 \text{ or } |\gamma| \geq 4/3, \\ s(\gamma - 1/2) & \text{if } \gamma \in (1/3, 2/3], \\ c(\gamma/2 - 1/2) & \text{if } \gamma \in (2/3, 4/3), \\ s(\gamma/2 + 1/2) & \text{if } \gamma \in (-4/3, -2/3), \\ c(\gamma + 1/2) & \text{if } \gamma \in [-2/3, -1/3). \end{cases}$$