Orthonormal Bases in Hilbert Space.

Linear (Vector) Spaces.

Definition 0.1 *A* **linear space** *is a nonempty set L together with a mapping from $L \times L$ into L called* addition*, denoted $(x, y) \longmapsto x + y$ and a mapping from the Cartesian product of either* $\mathbf{R}$ *or* $\mathbf{C}$ *with L into L called* scalar multiplication*, denoted $(\alpha, x) \longmapsto \alpha x$, which satisfy the following properties.*

(1) *Axioms of addition.*

  (a) $x + y = y + x$ *(commutativity).*

  (b) $(x + y) + z = x + (y + z)$ *(associativity).*

  (c) *There exists an element $0 \in L$, called the zero element, such that for all $x \in L$, $0 + x = x + 0 = x$.*

  (d) *For every $x \in L$, there is an element $-x \in L$ such that $x + (-x) = (-x) + x = 0$.*

(2) *Axioms of scalar multiplication.*

  (a) $\alpha(\beta x) = (\alpha\beta)x$*, for all scalars α, β and all $x \in L$.*

  (b) $1(x) = x$.

(3) *Distributive Laws.*

  (a) $(\alpha + \beta)x = \alpha x + \beta x$*, for all scalars α, β and all $x \in L$.*

  (b) $\alpha(x + y) = \alpha x + \alpha y$*, for all scalars α and all x, $y \in L$.*

Examples. (a) $L = \mathbf{R}^n$ or $\mathbf{C}^n$ with addition and scalar multiplication defined componentwise in the usual way.
(b) $L = C[0, 1]$, the space of functions continuous on $[0, 1]$, with addition of functions defined by $(f + g)(x) = f(x) + g(x)$, and scalar multiplication defined by $(\alpha f)(x) = \alpha f(x)$.
(c) The following spaces are examples of *sequence spaces* and are in some sense the natural generalizations of $\mathbf{R}^n$ as $n \to \infty$.

$$\begin{aligned}
\ell^\infty &= \{\{x_n\}_{n=1}^\infty : \sup |x_n| < \infty\} \\
c_0 &= \{\{x_n\}_{n=1}^\infty : \lim x_n = 0\} \\
\ell^2 &= \{\{x_n\}_{n=1}^\infty : \sum |x_n|^2 < \infty\} \\
\ell^1 &= \{\{x_n\}_{n=1}^\infty : \sum |x_n| < \infty\} \\
f &= \{\{x_n\}_{n=1}^\infty : \exists N = N(x), x_n = 0 \forall n \geq N\},
\end{aligned}$$

where $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\alpha\{x_n\} = \{\alpha x_n\}$.
Exercise 1. Show that $f \subset \ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty$ as sets.

Definition 0.2 *A subset L' of a linear space L is a* subspace *of L provided that for every x, $y \in L'$ and scalars α, β,*

$$\alpha x + \beta y \in L'.$$

In other words, L' is a subspace provided that it is a linear space in its own right with addition and scalar multiplication inherited from L.

Given a nonempty subset S of a linear space L, the linear span *or* span *of S, denoted* $\mathrm{span}(S)$*, is the set of all finite linear combinations of elements in S, that is*

$$\mathrm{span}(S) = \{\sum_{i=1}^{n} \alpha_i\, x_i \colon \alpha_i \text{ scalars},\ x_i \in S\}.$$

Proposition 0.1 *Given a nonempty subset S of a linear space L,* $\mathrm{span}(S)$ *is a subspace of L and moreover,* $\mathrm{span}(S)$ *is the smallest subspace of L containing S, that is, if L' is any subspace containing S, then* $\mathrm{span}(S) \subseteq L'$.

Proof: Exercise 2.

Normed Linear Spaces.

Definition 0.3 *A* **normed linear space** *is a pair $(V, \|\cdot\|)$ where V is a linear space (over* **R** *or* **C***), and $\|\cdot\|$ is a function $\|\cdot\| \colon V \longrightarrow \mathbf{R}$ called a* **norm** *which satisfies for all v, $w \in V$ the following properties.*

(1) $\|v\| \geq 0$.

(2) $\|v\| = 0$ *if and only if* $v = 0$.

(3) $\|\alpha v\| = |\alpha| \|v\|$ *for every scalar* α.

(4) $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality).*

Remark 0.1 Given a normed linear space V, we can define a metric on V by $d(x, y) = \|x - y\|$, making (V, d) a metric space.

Examples.

1. $(C[0,1], \|\cdot\|_\infty)$, where $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.
2. $(C[0,1], \|\cdot\|_1)$, where $\|f\|_1 = \int_0^1 |f(x)|\, dx$.
3. $(C[0,1], \|\cdot\|_2)$, where $\|f\|_2 = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}$.
4. $(\mathbf{R}^n, \|\cdot\|_2)$, where $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.
5. $(\mathbf{R}^n, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{i=1}^n |x_i|$.

6. $(\mathbf{R}^n, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_{1=1,\ldots,n} |x_i|$.

7. ℓ^1 where $\|\{x_n\}\|_1 = \sum_n |x_n|$.

8. ℓ^2 where $\|\{x_n\}\|_2 = (\sum_n |x_n|^2)^{1/2}$.

9. ℓ^∞ where $\|\{x_n\}\|_\infty = \sup_n |x_n|$.

Exercise 3. Show that for all $f \in C[0,1]$,

(a) $\|f\|_1 \leq \|f\|_\infty$

(b) $\|f\|_2 \leq \|f\|_\infty$

(c) $\|f\|_1 \leq \|f\|_2$ (Use the Cauchy-Schwarz inequality proved later.)

Show that in each case an opposite inequality does not hold in the sense that (for example) there is no number $C > 0$ such that for all $f \in C[0,1]$, $\|f\|_\infty \leq C\,\|f\|_1$. In other words, for each number C, find a function $f \in C[0,1]$ (which will depend on C) such that $\|f\|_\infty > C\,\|f\|_1$. Find corresponding examples for (b) and (c).

Exercise 4. Show that for all $x \in \mathbf{C}^n$,

(a) $\|x\|_\infty \leq \|x\|_1$

(b) $\|x\|_\infty \leq \|x\|_2$

(c) $\|x\|_2 \leq \|x\|_1$

(d) $\|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$

(e) $\|x\|_1 \leq n\,\|x\|_\infty$.

Use these inequalities to show that the three norms are equivalent.

Exercise 5. Show that for all complex-valued sequences $x = \{x_n\}_{n=1}^\infty$

(a) $\|x\|_\infty \leq \|x\|_1$

(b) $\|x\|_\infty \leq \|x\|_2$

(c) $\|x\|_2 \leq \|x\|_1$.

Show that in each case an opposite inequality does not hold in the same sense as in Exercise 3. That is, show that (for example) given $C > 0$ there is a sequence $x = \{x_n\}$ (which will depend on C) such that $\|x\|_1 > C\,\|x\|_\infty$.

Inner Product and Hilbert Spaces.

Definition 0.4 *An* **inner product space** *is a pair* $(V, \langle\cdot,\cdot\rangle)$ *where* V *is a vector space over* $\mathbf{C}$ *or* $\mathbf{R}$ *and where* $\langle\cdot,\cdot\rangle$ *is a complex valued function*

$$\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathbf{C}$$

called the **inner product** *on* V *satisfying the following properties. For all* x, y, $z \in V$ *and* $\alpha \in \mathbf{C}$,

(1) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.

(3) $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$.

(4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Examples.

1. $\mathbf{R}^n$ with $\langle x, y\rangle = \sum_{i=1}^{n} x_i\, y_i$.
2. $V = C[0, 1]$ with $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\, dx$.
3. $V = \ell^2$ with $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n\, \overline{y_n}$.

Theorem 0.1 *Let* V *be an inner product space. Define the function*

$$\|\cdot\| : V \longrightarrow \mathbf{R}$$

by $\|x\| = \langle x, x\rangle^{1/2}$. *Then* $\|\cdot\|$ *is a norm on* V *making* V *a normed linear space.*

Corollary 0.1 (Cauchy-Schwarz Inequality). *For* x, $y \in V$, *an inner product space,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Definition 0.5 *A* **Hilbert space** *is a complete inner product space. (*Complete *here means that every Cauchy sequence converges.*

Examples.

1. $L^2[0,1]$ with the inner product given by
$$\langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\,dx$$
is a Hilbert space.

2. ℓ^2 with the inner product given by
$$\langle \{x_n\}, \{y_n\}\rangle = \sum_n x_n\,\overline{y_n}$$
is a Hilbert space.

3. $L^2(\mathbf{R})$ with the inner product given by
$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$$
is a Hilbert space.

Exercise 6. Show that the sequence space f of finite sequences is *not* complete by finding a sequence in f which is Cauchy but which does not converge to anything in f.

Orthonormal systems in Hilbert spaces.

Definition 0.6 *Let V be an inner product space. Two vectors $x, y \in V$ are said to be* **orthogonal** *if $\langle x, y\rangle = 0$. We also write in this case $x \perp y$.*

A collection of vectors $\{x_\alpha\}_{\alpha\in A} \subseteq V$ is said to be an **orthonormal system** *if $\langle x_\alpha, x_\beta\rangle = 0$ for $\alpha \neq \beta$ and if $\langle x_\alpha, x_\alpha\rangle = 1$ for all $\alpha \in A$.*

Lemma 0.1 (Best Approximation Lemma). *Let $\{x_n\}_{n=1}^N$ be an orthonormal system in an inner product space V and let $\{a_n\}_{n=1}^N$ be a finite sequence of scalars. Then for all $x \in V$,*
$$\left\| x - \sum_{n=1}^N a_n\,x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n\rangle|^2 + \sum_{n=1}^N |a_n - \langle x, x_n\rangle|^2.$$

Corollary 0.2 (1) $\displaystyle \left\| \sum_{n=1}^N a_n\,x_n \right\|^2 = \sum_{n=1}^N |a_n|^2.$

(2) $\displaystyle \left\| x - \sum_{n=1}^N \langle x, x_n\rangle\,x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n\rangle|^2.$

(3) (Bessel's Inequality) $\displaystyle \sum_{n=1}^N |\langle x, x_n\rangle|^2 \leq \|x\|^2.$

(4) *If $\{x_n\}_{n=1}^\infty$ is an orthonormal system, then for each $x \in V$, the series $\displaystyle\sum_{n=1}^\infty |\langle x, x_n\rangle|^2$ converges and $\displaystyle\sum_{n=1}^\infty |\langle x, x_n\rangle|^2 \leq \|x\|^2$.*

Orthonormal bases in Hilbert spaces.

Definition 0.7 *A collection of vectors* $\{x_\alpha\}_{\alpha\in A}$ *in a Hilbert space* H *is* **complete** *if* $\langle y, x_\alpha\rangle = 0$ *for all* $\alpha \in A$ *implies that* $y = 0$.

An equivalent definition of completeness is the following. $\{x_\alpha\}_{\alpha\in A}$ *is complete in* V *if* $\mathrm{span}\{x_\alpha\}$ *is dense in* V*, that is, given* $y \in H$ *and* $\epsilon > 0$*, there exists* $y' \in \mathrm{span}\{x_\alpha\}$ *such that* $\|x - y\| < \epsilon$*. Another way to put this is that given* y*, every ball around* y *contains an element of* $\mathrm{span}\{x_\alpha\}$*. The proof of this equivalence relies on a fundamental decomposition property of Hilbert spaces.*

An **orthonormal basis** *a complete orthonormal system.*

Theorem 0.2 *Let* $\{x_n\}_{n=1}^\infty$ *be an orthonormal system in a Hilbert space* H*. Then the following are equivalent.*

(1) $\{x_n\}$ *is complete.*

(2) *For all* $x \in H$*,*

$$x = \sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n,$$

where the sum converges unconditionally, that is, regardless of order.

(3) $\|x\|^2 = \sum_{n=1}^\infty |\langle x, x_n\rangle|^2$.

(4) *For all* x, $y \in H$*,*

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, x_n\rangle \overline{\langle y, x_n\rangle}.$$

Examples.

1. $\mathbf{R}^n$ or $\mathbf{C}^n$. In finite dimensional vector spaces we have the notion of *linear independence* and *dimension.* Specifically if the finite dimensional vector space X has dimension N and if $V = \{v_k\}_{k=1}^N$ is an orthonormal system, then it is an orthonormal basis. Any collection of N linearly independent vectors can be orthogonalized via the Gram-Schmidt process into an orthonormal basis.

2. $L^2[0,1]$ is the space of all Lebesgue measurable functions on $[0,1]$, square-integrable in the sense of Lebesgue. This space can be thought of as the completion of the incomplete normed linear space $C[0,1]$ with respect to the norm $\|f\|_2^2 = \int_0^1 |f(x)|^2\, dx$. In other words, each element of $L^2[0,1]$ can be thought of as the limit (in the sense of the L^2 norm) of a Cauchy sequence of continuous functions.

   The trigonometric system $\{e^{2\pi inx}\}_{n=-\infty}^\infty$ is an orthonormal basis for $L^2[0,1]$. The expansion of a function in this basis is called the *Fourier series* of that function.

   Another example of an orthonormal basis for $L^2[0,1]$ are the Legendre polynomials which are obtained by taking the sequence of monomials $\{1, x, x^2, \ldots\}$ and applying the Gram-Schmidt orthogonalization process to it.

3. $L^2(\mathbf{R})$ is the space of all Lebesgue measurable functions on $\mathbf{R}$, square-integrable in the sense of Lebesgue. This space can be thought of as the completion of the incomplete normed linear space $C_c(\mathbf{R})$ of functions continuous on $\mathbf{R}$ with compact support (equipped with the L^2 norm), or as the completion of the incomplete normed linear space $C_c^\infty(\mathbf{R})$ of all compactly supported, infinitely differentiable functions on $\mathbf{R}$ equipped again with the L^2 norm. This will be our setting for much of our discussion of wavelet bases. We will show how to construct orthonormal bases of this space with wavelets.

Exercise 7. Show that the trigonometric system $\{e^{2\pi inx}\}_{n=-\infty}^{\infty}$ is an orthonormal system in $L^2[0,1]$.

Exercise 8. Find the first four Legendre polynomials by applying the Gram-Schmidt process to the sequence $\{1,\, x,\, x^2,\, \ldots\}$.