Review of Fourier Analysis.

A. The Fourier Transform.

Definition 0.1 *The normed linear space* $L^1(\mathbf{R})$ *consists of all functions* f*, Lebesgue measurable on* $\mathbf{R}$ *with the property that the norm*

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx$$

is finite.

Remarks. (a) $L^1(\mathbf{R})$ is a linear space under the usual addition and scalar multiplication of functions.
(b) We saw before that $L^2[0,1] \subseteq L^1[0,1]$ which follows from the inequality $\|f\|_1 \leq \|f\|_2$ (where the norms are on the spaces $L^r[0,1]$, $r = 1,\ 2$). However this is false if $[0,1]$ is replaced by $\mathbf{R}$.
(c) The spaces $C_c(\mathbf{R})$ and $C_c^\infty(\mathbf{R})$ are both dense in $L^1(\mathbf{R})$.

Definition 0.2 *The* Fourier transform *of* $f \in L^1(\mathbf{R})$*, denoted* $\widehat{f}(\gamma)$*, is given by,*

$$\widehat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\gamma x}\, dx.$$

Remarks. (a) Note a superficial resemblance to the definition of the Fourier coefficients of a function on $[0,1]$, namely

$$\widehat{f}(n) = \int_{0}^{1} f(x)\, e^{-2\pi i n x}\, dx.$$

We will explore this relationship later.
(b) We assume that $f \in L^1(\mathbf{R})$ in order to guarantee that the integral converges. This is only technical as the integral can be interpreted for a variety of function spaces.

Definition 0.3 *The* Fourier inversion formula *is the following. If* $\widehat{f}(\gamma)$ *is the Fourier transform of* $f(x)$ *then,*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\gamma)\, e^{2\pi i\gamma x}\, d\gamma.$$

Remarks. (a) Fourier inversion is not always valid. Indeed a major focus of a rigorous course in Fourier analysis is to determine conditions under which the inversion formula holds pointwise and in other senses.
(b) We will give the precise theorem later but will typically use Fourier inversion without regard for the precise nature of its validity.

Proposition 0.1 *Suppose that* $f \in L^1(\mathbf{R})$*. Then* $\widehat{f} \in C_0(\mathbf{R})$*, that is,* $\widehat{f}$ *(a) is continuous on the real line and (b) converges to zero at infinity.*

Remarks. (a) Note that $f \in L^1(\mathbf{R})$ does not necessarily imply that $\widehat{f} \in L^1$. In fact, an easy counterexample to that is the function $f(x) = \chi_{[-1/2,1/2]}(x)$. (See Exercise 3.4 in Walnut)

(b) It is possible to manufacture functions $f \in L^1(\mathbf{R})$ such that $\widehat{f}(\gamma)$ converges to zero as slowly as desired.

Theorem 0.1 *If both f and $\widehat{f} \in L^1(\mathbf{R})$, then the Fourier inversion formula holds at each point x. Note that any such f for which this holds pointwise must be also in $C_0(\mathbf{R})$).*

B. Invariance Properties of the Fourier Transform.

(1) Translation, Dilation, and Modulation.

Definition 0.4 *Given $a > 0$, the* dilation operator*, D_a is given by*

$$D_a f(x) = a^{1/2} f(ax).$$

Given $b \in \mathbf{R}$, the translation operator*, T_b is given by*

$$T_b f(x) = f(x - b).$$

Given $c \in \mathbf{R}$, the modulation operator*, M_c is given by*

$$M_c f(x) = e^{2\pi icx} f(x).$$

Theorem 0.2 *Let $f \in L^1(\mathbf{R})$.*
(a) *For every $a > 0$, $\widehat{D_a f}(\gamma) = D_{1/a}\widehat{f}(\gamma)$.*
(b) *For every $b \in \mathbf{R}$, $\widehat{T_b f}(\gamma) = M_{-b}\widehat{f}(\gamma)$.*
(c) *For every $c \in \mathbf{R}$, $\widehat{M_c f}(\gamma) = T_c\widehat{f}(\gamma)$.*

Theorem 0.3 (Translation and Dilation) *For every f, $g \in L^2(\mathbf{R})$, and for every $a > 0$, $b \in \mathbf{R}$,*
(c) $\langle f, D_a g\rangle = \langle D_{a^{-1}} f, g\rangle$.
(d) $\langle f, T_b g\rangle = \langle T_{-b} f, g\rangle$.
(e) $\langle f, D_a T_b g\rangle = \langle T_{-b} D_{a^{-1}} f, g\rangle$.
(f) $\langle D_a f, D_a g\rangle = \langle f, g\rangle$.
(g) $\langle T_b f, T_b g\rangle = \langle f, g\rangle$.

Theorem 0.4 (Translation and Modulation) *For every f, $g \in L^2(\mathbf{R})$, and for every b, $c \in \mathbf{R}$,*
(a) $T_b M_c f(x) = e^{-2\pi ibc} M_c T_b f(x)$.
(b) $\langle f, M_c g\rangle = \langle E_{-c} f, g\rangle$.
(c) $\langle f, T_b M_c g\rangle = e^{2\pi ibc} \langle T_{-b} M_{-c} f, g\rangle$.

(2) Differentiation.

Theorem 0.5 (Differentiation Theorem) *If f, $x\,f(x) \in L^1(\mathbf{R})$, then $\widehat{f} \in C^1(\mathbf{R})$, and*

$$\widehat{xf}(\gamma) = \frac{-1}{2\pi i}\,\frac{d\widehat{f}}{d\gamma}(\gamma).$$

Corollary 0.1 *If f, $x^N f(x) \in L^1(\mathbf{R})$ for some $N \in \mathbf{N}$, then $\widehat{f} \in C^N(\mathbf{R})$, and for $0 \le j \le N$,*

$$\widehat{x^j f}(\gamma) = \left(\frac{-1}{2\pi i}\right)^j \frac{d^j \widehat{f}}{d\gamma^j}(\gamma).$$

(3) Convolution and Involution.

Definition 0.5 *Given functions $f(x)$ and $g(x)$, the* convolution *of $f(x)$ and $g(x)$, denoted $h(x) = f * g(x)$, is defined by*

$$f * g(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

Remarks. (a) The convolution $f * g(x)$ can be interpreted as a "moving weighted average" of $f(x)$, where the "weighting" is determined by the function $g(x)$.

(b) Convolution is *commutative*, i.e., $f * g(x) = g * f(x)$.

(c) Convolution is a *smoothing operation.* In general, $f * g(x)$ will be *at least as smooth* as the smoothest of f and g, and if both f and g are smooth, it will pick up the smoothness of both.

(d) The convolution of f and g will in general decay at infinity *only as fast as the slowest-decaying of f or g.*

Theorem 0.6 (The Convolution Theorem) *If f, $g \in L^1(\mathbf{R})$, then*

$$\widehat{f * g}(\gamma) = \widehat{f}(\gamma)\,\widehat{g}(\gamma).$$

Definition 0.6 *Given $f(x)$, the* involution *of f, denoted $f^*(x)$, is defined by $f^*(x) = \overline{f(-x)}$.*

Theorem 0.7

$$\widehat{f^*}(\gamma) = \overline{\widehat{f}(\gamma)}.$$

C. Basic Properties of the Fourier Transform.

(1) Smoothness versus Decay.

A fundamental principle of use in interpreting many results about the Fourier transform is the following: *Smooth functions have Fourier transforms that decay rapidly to zero at infinity, and functions that decay rapidly to zero at infinity have smooth Fourier transforms.*

Theorem 0.8 (Differentiation Theorem) *If f, $x\, f(x) \in L^1(\mathbf{R})$, then $\widehat{f} \in C^1(\mathbf{R})$, and*

$$\widehat{xf}(\gamma) = \frac{-1}{2\pi i}\,\frac{d\widehat{f}}{d\gamma}(\gamma).$$

Corollary 0.2 *If f, $x^N\, f(x) \in L^1(\mathbf{R})$ for some $N \in \mathbf{N}$, then $\widehat{f} \in C^N(\mathbf{R})$, and for $0 \le j \le N$,*

$$\widehat{x^j f}(\gamma) = \left(\frac{-1}{2\pi i}\right)^j \frac{d^j \widehat{f}}{d\gamma^j}(\gamma).$$

Corollary 0.3 *If f, $\widehat{f}$, $\gamma^N\, \widehat{f}(\gamma) \in L^1(\mathbf{R})$, then $f \in C^N(\mathbf{R})$, and for $0 \le j \le N$,*

$$f^{(j)}(x) = \int_{-\infty}^{\infty} (2\pi i\gamma)^j\, \widehat{f}(\gamma)\, e^{2\pi i\gamma x}\, d\gamma.$$

Theorem 0.9 *Suppose that $f \in L^1(\mathbf{R})$, and that for some $N \in \mathbf{N}$,*
(1) $\widehat{f} \in C^N(\mathbf{R})$,
(2) $\widehat{f}$, $\widehat{f}^{(N)} \in L^1(\mathbf{R})$
(3) For $0 \le j \le N$, $\lim_{|\gamma|\to\infty} \widehat{f}^{(j)}(\gamma) = 0$. Then

$$\lim_{|x|\to\infty} x^N\, f(x) = 0.$$

(2) Plancherel's, Parseval's, and Poisson's Summation formula.

Theorem 0.10 (Plancherel's Formula) *If $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\widehat{f} \in L^2(\mathbf{R})$ and*

$$\int_{-\infty}^{\infty} |\widehat{f}(\gamma)|^2\, d\gamma = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

Theorem 0.11 (Parseval's Formula) *If f, $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, then*

$$\int_{-\infty}^{\infty} \widehat{f}(\gamma)\, \overline{\widehat{g}(\gamma)}\, d\gamma = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx.$$

Theorem 0.12 (Poisson's Summation Formula) *If $f(x)$ is sufficiently smooth and has sufficiently rapid decay, then*

$$\sum_n f(x+n) = \sum_m \widehat{f}(m)\, e^{2\pi i m x}.$$

Corollary 0.4 *If f is such that the PSF holds, and if $\delta > 0$ and $\alpha \in \mathbf{R}$, then*

$$\sum_{n=-\infty}^{\infty} f\left(\frac{x+n}{\delta}\right) e^{-2\pi i n\alpha/\delta} = \delta \sum_{m=-\infty}^{\infty} \widehat{f}(\delta m + \alpha)\, e^{2\pi i x(\delta m+\alpha)/\delta}.$$

Definition 0.7 *A function $f \in L^2(\mathbf{R})$, is* bandlimited *if there is a number $\Omega > 0$ such that $\widehat{f}(\gamma)$ is supported in the interval $[-\Omega/2, \Omega/2]$. In this case, the function $f(x)$ is said to have* bandlimit *$\Omega > 0$.*

Theorem 0.13 (The Shannon Sampling Theorem) *If $f(x)$ is bandlimited with bandlimit Ω, then $f(x)$ can be written as*

$$f(x) = \sum_n f(n/\Omega) \frac{\sin(\pi\Omega(x - n/\Omega))}{\pi\Omega(x - n/\Omega)},$$

in L^2 and L^∞ on $\mathbf{R}$.

D. The Uncertainty Principle and the Time-Frequency Plane.

Suppose that $f \in L^2(\mathbf{R})$ and that $\|f\|_2 = 1$. Then we can consider the function $|f(x)|^2$ to be a probability distribution with expected value

$$\bar{x} = \int_{-\infty}^{\infty} x\,|f(x)|^2\,dx$$

and standard deviation

$$\Delta_f x = \Big(\int_{-\infty}^{\infty} (x - \bar{x})^2 |f(x)|^2\,dx\Big)^{1/2}.$$

By Plancherel's formula, $\widehat{f} \in L^2(\mathbf{R})$ also satisfies $\|\widehat{f}\|_2 = 1$ and we write

$$\bar{\gamma} = \int_{-\infty}^{\infty} \gamma\,|\widehat{f}(\gamma)|^2\,d\gamma$$

and

$$\Delta_f \gamma = \Big(\int_{-\infty}^{\infty} (\gamma - \bar{\gamma})^2 |\widehat{f}(\gamma)|^2\,d\gamma\Big)^{1/2}$$

for the expected value and standard deviation of the probability distribution $|\widehat{f}(\gamma)|^2$.

Theorem 0.14 Classical Uncertainty Principle. *If $f \in L^2(\mathbf{R})$ then*

$$\Delta_f x \cdot \Delta_f \gamma \geq \frac{1}{4\pi}$$

and this inequality is minimized when $f(x) = e^{2\pi i\bar{\gamma}(x-\bar{x})}\,e^{-\pi(x-\bar{x})^2/c} = T_{\bar{x}} M_{\bar{\gamma}}(e^{-\pi(\cdot)^2/c})(x)$ for some $c > 0$.

Remark. (a) Roughly speaking we can say that $\Delta_f x$ measures the "essential support" of $f(x)$ in the sense that the function $f(x)$ is thought of as "mostly concentrated" in an interval of length $2\Delta_f x$. Similarly, $\Delta_f \gamma$ measures the essential support of $\widehat{f}(\gamma)$. Hence the uncertainty principle says that a function and its Fourier transform cannot both be well-concentrated around their respective means.

(b) This inequality is formulated as: *A realizable signal occupies a region of area at least one in the time-frequency plane.* A more precise formulation of this principle occurs in the following inequality of Donoho and Stark.

Definition 0.8 *A function* $f \in L^2(\mathbf{R})$ *is* ϵ-concentrated *on a set* T *if*

$$\left(\int_{T^c} |f(x)|^2 \, dx \right)^{1/2} < \epsilon \, \|f\|_2.$$

Theorem 0.15 *Suppose that* $f \in L^2(\mathbf{R})$ *is* ϵ_T*-concentrated on the set* $T \subseteq \mathbf{R}$ *and* $\widehat{f}$ *is* ϵ_Ω*-concentrated on the set* $\Omega \subseteq \mathbf{R}$*. Then*

$$|T| \, |\Omega| \geq (1 - \epsilon_T - \epsilon_\Omega)^2.$$