Local Trigonometric Bases.

The goal is to develop an orthonormal basis for $L^2(\mathbf{R})$ whose elements are localized to a tiling of the time-frequency plane corresponding to an arbitrary partition of the *time-axis*. This is to be contrasted to wavelet-packet bases that are localized to a tiling of the time-frequency plane corresponding to an arbitrary dyadic partition of the *frequency-axis*.

A. Trigonometric Bases on Intervals.

Theorem 1 *Let $I = [a, a+l]$. Then the following collections are orthonormal bases for $L^2(I)$.*

(a) $\left\{\sqrt{\dfrac{2}{l}}\cos\dfrac{2k+1}{2l}\pi(x-a)\right\}_{k=0}^{\infty}$.

(b) $\left\{\sqrt{\dfrac{2}{l}}\sin\dfrac{2k+1}{2l}\pi(x-a)\right\}_{k=0}^{\infty}$.

(c) $\left\{\sqrt{\dfrac{2}{l}}\sin\dfrac{k}{l}\pi(x-a)\right\}_{k=1}^{\infty}$.

(d) $\left\{\sqrt{\dfrac{2}{l}}\cos\dfrac{k}{l}\pi(x-a)\right\}_{k=0}^{\infty}$.

The proof of this theorem relies on the following lemma.

Lemma 1 *Let $I = [c-b, c+b]$. Then the collection*

$$\left\{\frac{1}{\sqrt{2b}},\ \frac{1}{\sqrt{b}}\cos\frac{k}{b}\pi(x-c),\ \frac{1}{\sqrt{b}}\sin\frac{k}{b}\pi(x-c)\right\}_{k=1}^{\infty}$$

is an orthonormal basis for $L^2(I)$.

The idea of the proof of the Theorem is to start with a function defined on $[a, a+l]$, extend it to be odd or even about a, resulting in a function defined on $[a-l, a+l]$, then extend that function to be odd or even about $a+l$ and $a-l$ (same symmetry about each point), and consider the portion of this function defined on $[a-2l, a+2l]$. Expand this function in the basis given in the Lemma. The symmetries imply that most of the coefficients will vanish except those attached to the elements of the collections given in the Theorem. Each such collection corresponds to a choice of symmetries about a and $a+l$ (respectively) as follows.

(a) $\left\{\sqrt{\dfrac{2}{l}}\cos\dfrac{2k+1}{2l}\pi(x-a)\right\}_{k=0}^{\infty}$ (even/odd).

(b) $\left\{\sqrt{\dfrac{2}{l}}\sin\dfrac{2k+1}{2l}\pi(x-a)\right\}_{k=0}^{\infty}$ (odd/even).

(c) $\left\{\sqrt{\frac{2}{l}}\,\sin\frac{k}{l}\pi(x-a)\right\}_{k=1}^{\infty}$ (odd/odd).

(d) $\left\{\sqrt{\frac{2}{l}}\,\cos\frac{k}{l}\pi(x-a)\right\}_{k=0}^{\infty}$ (even/even).

B. Smooth bell functions over intervals.

Recall that in the construction of the Meyer wavelet, we encountered smooth functions $s_\epsilon(x)$ and $c_\epsilon(x)$ (for any given $\epsilon > 0$ with the property that

(1) $s_\epsilon(x) = c_\epsilon(-x)$,

(2) $s_\epsilon^2(x) + c_\epsilon^2(x) = 1$,

(3) $0 \le s_\epsilon(x),\ c_\epsilon(x) \le 1$,

(4) $s_\epsilon(x) = 0$ for $x \le -\epsilon$ and 1 for $x \ge \epsilon$, and

(5) $c_\epsilon(x) = 0$ for $x \ge \epsilon$ and 1 for $x \le -\epsilon$.

Definition 1 *Given $I = [\alpha, \beta]$ and ϵ, $\epsilon' > 0$ such that $\alpha + \epsilon \le \beta - \epsilon'$, define the function $b_I(x)$ by $b_I(x) = s_\epsilon(x-\alpha)\, c_{\epsilon'}(x-\beta)$. $b_I(x)$ is referred to as a* smooth bell function *over I.*

Definition 2 *Given adjacent intervals $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$, the smooth bell functions b_I and b_J are* compatible *if $b_I(x) = s_\epsilon(x-\alpha)\, c_{\epsilon'}(x-\beta)$ and $b_J(x) = s_{\epsilon'}(x-\alpha)\, c_{\epsilon''}(x-\beta)$.*

C. Smooth Projections.

Definition 3 *Given I, ϵ, ϵ' as above, define the operator P_I on $L^2(\mathbf{R})$ by*

$$P_I f(x) = b_I(x)[b_I(x)f(x) \pm b_I(2\alpha - x)f(2\alpha - x) \pm b_I(2\beta - x)f(2\beta - x)].$$

Remarks.

1. Note that $P_I f(x)$ is supported in the interval $[\alpha - \epsilon, \beta + \epsilon']$.
2. $\pm$ means that we can make either choice in each place where this occurs. This means that there are four flavors of P_I, $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$. The first $\pm$ is called the *polarity of P_I at α* and the second $\pm$ is called the *polarity of P_I at β*.
3. $f(2\alpha - x)$ is the reflection of $f(x)$ about the line $x = \alpha$, similarly for $f(2\beta - x)$.
4. $P_I f(x)$ is obtained by (a) cutting off $f(x)$ by multiplication by $b_I(x)$, (b) extending the result to be even or odd about α (corresponding to the first $\pm$, $+ =$ *even*, $- =$ *odd*), then extending that result to be even or odd about β (corresponding to the first $\pm$), (c) cutting off the result by multiplication by $b_I(x)$ again.

5. We write $P_I f(x) = b_I(x)\, S(x)$ where $S(x)$ is even/odd about α and even/odd about β depending on the polarity of P_I at those points.

Theorem 2 *The operators P_I satisfy the following properties.*

(a) *P_I is an orthogonal projector. This means that $P_I^2 = P_I$ and $P_I^* = P_I$.*

(b) *If b_I and b_J are compatible bells on the intervals $I = \alpha, \beta]$ and $J = [\beta, \gamma]$ and if P_I and P_J have opposite polarities at β then $P_I + P_J = P_{I \cup J}$.*

(c) *In the case of (b) above, $P_I\, P_J = 0$.*

Theorem 3 *Choose a sequence $\{\alpha_j\}_{j \in \mathbf{Z}}$ strictly increasing and going to infinity in both directions and numbers ϵ_j satisfying $\alpha_j + \epsilon_j \leq \alpha_{j+1} - \epsilon_{j+1}$. Note that this ensures that there exist compatible smooth bell functions over each interval I_j. Now choose smooth projections P_j onto $I_j = [\alpha_j, \alpha_{j+1}]$ with opposite polarities at the common endpoints. Then the projectors $\{P_j\}$ split $L^2(\mathbf{R})$ into mutually orthogonal subspaces, $H_j = P_j(L^2(\mathbf{R}))$.*

D. Orthogonal Expansions of $H_j = P_j(L^2(\mathbf{R}))$.

Recall that the orthonormal bases for intervals given in the first Theorem correspond to the expansions in an ordinary sine and cosine basis of the even or odd extensions of functions about the endpoints of the interval in their various combinations.

Now fix an interval $I = [\alpha, \beta]$ and numbers ϵ and ϵ' as before, and a polarity (i.e. a choice of $+$ or $-$) for a projection P_I. Finally let $\{e_k(x)\}$ denote the sine or cosine basis from the first Theorem with the same polarity. Then the following holds.

Theorem 4 *If I is given as above and if b_I is a smooth bell function over I, then the collection $\{b_I(x)\, e_k(x)\}_k$ is an orthonormal basis for the subspace $P_I(L^2(\mathbf{R}))$.*

For the proof of this theorem we must verify three things.

(a) $\{b_I\, e_k\}$ is an orthonormal system in $L^2(\mathbf{R})$.

(b) For all $f \in L^2$, $P_I f(x) = \sum_k c_k\, b_I(x)\, e_k(x)$ for some coefficients $\{c_k\}$ with $\sum_k |c_k|^2 < \infty$.

(c) $b_I\, e_k \in P_I(L^2(\mathbf{R}))$ for all k.

For part (c) the following Lemma is sufficient and also will be useful later on.

Lemma 2 *Suppose that $g \in L^2(\mathbf{R})$ is even or odd about α and β in some combination, and that the smooth projector P_I has the same polarity at α and β. Then $P_I(b_I\, g) = b_I\, g$.*

E. Local Trigonometric Bases for $L^2(\mathbf{R})$.

Theorem 5 *Choose a sequence $\{\alpha_j\}_{j\in\mathbf{Z}}$ strictly increasing and going to infinity in both directions and numbers ϵ_j satisfying $\alpha_j + \epsilon_j \leq \alpha_{j+1} - \epsilon_{j+1}$. Now choose smooth projections P_j onto $I_j = [\alpha_j, \alpha_{j+1}]$ with opposite polarities at the common endpoints. Let $\theta_{j,k}(x) = b_{I_j}(x)\, e_{j,k}(x)$ where $\{e_{j,k}(x)\}_k$ is the sine or cosine basis on $L^2(I_j)$ with the same polarity as P_j at each endpoint. Then $\{\theta_{j,k}\}$ is an orthonormal basis for $L^2(\mathbf{R})$.*

The proof of this theorem requires that we verify orthogonality and completeness. Orthogonality for $\theta_{j,k}$ corresponding to nonadjacent intervals follows since their supports are disjoint. Orthogonality for functions corresponding to the same interval follows from the previous Theorem. Orthogonality for functions corresponding to adjacent intervals follows from the previous Lemma and a property of smooth projections.

For example, choosing the polarity $(-,+)$ for all of the P_j gives the basis

$$\theta_{j,k}(x) = \sqrt{\frac{2}{|I_j|}}\, b_{I_j}(x)\, \sin \frac{2k+1}{2|I_j|}\pi(x-\alpha_j), \quad j \in \mathbf{Z},\ k \geq 0.$$

Choosing the polarity $(+,-)$ for each P_j gives the basis

$$\theta_{j,k}(x) = \sqrt{\frac{2}{|I_j|}}\, b_{I_j}(x)\, \cos \frac{2k+1}{2|I_j|}\pi(x-\alpha_j), \quad j \in \mathbf{Z},\ k \geq 0.$$

Note that the polarities of the P_j can be mixed and matched as desired as long as the appropriate basis is chosen on each I_j.