The Daubechies Wavelets.

A. Vanishing Moments.

Related to smoothness:

**Theorem 0.1** Let $\psi(x)$ be such that for some $N \in \mathbb{N}$, both $x^N \psi(x)$ and $\gamma^{N+1} \hat{\psi}(\gamma)$ are in $L^1(\mathbb{R})$. If $\{\psi_{j,k}(x)\}_{j,k \in \mathbb{Z}}$ is an orthogonal system on $\mathbb{R}$, then $\int_{\mathbb{R}} x^m \psi(x) \, dx = 0$ for $0 \leq m \leq N$.

Related to approximation of smooth functions:

**Theorem 0.2** Given $N \in \mathbb{N}$, assume that the function $f \in C^N(\mathbb{R})$, and that $f^{(N)} \in L^\infty(\mathbb{R})$. Assume that the function $\psi(x)$ has compact support, that $\int_{\mathbb{R}} x^m \psi(x) \, dx = 0$, for $0 \leq m \leq N-1$ and that $\int_{\mathbb{R}} |\psi_{j,k}(x)|^2 \, dx = 1$ for all $j, k \in \mathbb{Z}$. Then there is a constant $C > 0$ depending only on $N$ and $f(x)$ such that for every $j, k \in \mathbb{Z}$, $|\langle f, \psi_{j,k} \rangle| \leq C 2^{-jN} 2^{-j/2}$.

Related to reproduction of polynomials:

**Theorem 0.3** Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA, and let $\psi(x)$ be the wavelet. If $\psi(x)$ has $N$ vanishing moments, then for each integer $0 \leq k \leq N-1$, there are coefficients $\{q_{k,n}\}_{n \in \mathbb{Z}}$ such that $\sum_n q_{k,n} \varphi(x+n) = x^k$.

Equivalent conditions for vanishing moments:

**Theorem 0.4** Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA with finite scaling filter $h(n)$. Let $\psi(x)$ be the corresponding wavelet. Then for each $N \in \mathbb{N}$, the following are equivalent.

(a) $\int_{\mathbb{R}} x^k \psi(x) \, dx = 0$ for $0 \leq k \leq N-1$.

(b) $m_0^{(k)}(1/2) = 0$, for $0 \leq k \leq N-1$.

(c) $m_0(\gamma)$ can be factored as $m_0(\gamma) = \left(\frac{1+e^{-2\pi i \gamma}}{2}\right)^N \mathcal{L}(\gamma)$ for some period 1 trigonometric polynomial $\mathcal{L}(\gamma)$.

(d) $\sum_n h(n) (-1)^n n^k = 0$ for $0 \leq k \leq N-1$.

B. Daubechies Polynomials.

(1) We want to construct a trig polynomial $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k) e^{-2\pi ik\gamma}$ satisfying $m_0(\gamma) = \left(\frac{1+e^{-2\pi i \gamma}}{2}\right)^N \mathcal{L}(\gamma)$, and satisfying the QMF conditions.
\( |m_0(\gamma)|^2 = \frac{1 + e^{-2\pi i \gamma}}{2}^{2N} |\mathcal{L}(\gamma)|^2 = \cos^{2N}(\pi \gamma) L(\gamma). \)

(3) Since \( L(\gamma) \) is a real-valued trig polynomial with real coefficients, we arrive at \( L(\gamma) = P(\sin^2(\pi \gamma)) \) for some polynomial \( P \).

(4) This polynomial \( P \) must satisfy \( 1 = (1 - y)^N P(y) + y^N P(1 - y) \) with \( P(y) \geq 0 \) for all \( 0 \leq y \leq 1 \).

(5) We arrive at finally the definition

\[
P_{N-1}(y) = \sum_{k=0}^{N-1} \left( \frac{2N - 1}{k} \right) y^k (1 - y)^{N-1-k}.
\]

For example,

\[
\begin{align*}
P_0(y) &= 1, \\
P_1(y) &= 1 + 2y, \\
P_2(y) &= 1 + 3y + 6y^2, \\
P_3(y) &= 1 + 4y + 10y^2 + 20y^3.
\end{align*}
\]

C. Some simple examples.

If \( m_0(\gamma) = \left( \frac{1 + e^{-2\pi i \gamma}}{2} \right)^N \mathcal{L}(\gamma) \) then \( |m_0(\gamma)|^2 = \cos^{2N}(\pi \gamma) P_{N-1}(\sin^2 \pi \gamma) \), so if we solve \( |\mathcal{L}(\gamma)|^2 = P_{N-1}(\sin^2 \pi \gamma) \) then we have \( m_0(\gamma) \) and hence our scaling sequence \( \{h(n)\} \).

(a) \( N = 1 \). \( P_0(y) = 1 \) so \( \mathcal{L}(\gamma) = 1 \) and

\[
m_0(\gamma) = \left( \frac{1 + e^{-2\pi i \gamma}}{2} \right) = \frac{1}{2} + \frac{1}{2} e^{-2\pi i \gamma} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e^{-2\pi i \gamma} \right)
\]

so \( h(0) = h(1) = 1/\sqrt{2} \) and \( h(n) = 0 \) otherwise and we have recovered the Haar system.

(b) \( N = 2 \). Here \( P_1(y) = 1 + 2y \) so that we must solve

\[
|\mathcal{L}(\gamma)|^2 = 1 + 2 \sin^2 \pi \gamma = 1 + 2((1 - \cos(2\pi \gamma))/2) = 2 - \cos(2\pi \gamma).
\]

If \( \mathcal{L}(\gamma) = a + b e^{-2\pi i \gamma} \) then

\[
|\mathcal{L}(\gamma)|^2 = (a^2 + b^2) + 2ab \cos(2\pi \gamma).
\]

This, together with the normalization \( m_0(0) = \mathcal{L}(0) = a + b = 1 \) leads to the nonlinear system

\[
\begin{align*}
a^2 + b^2 &= 2, \\
2ab &= -1, \\
a + b &= 1
\end{align*}
\]

which when solved gives the Daubechies 4–coefficient wavelet filter.

D. Spectral Factorization.
1. Look directly at the equation \( |m_0(\gamma)|^2 = \cos^{2N}(\pi \gamma) P_{N-1}(\sin^2(\pi \gamma)) \).

2. Extend to the complex plane by making the substitution \( z = e^{2\pi i \gamma} \). Then

\[
\cos(2\pi \gamma) = \frac{e^{2\pi i \gamma} + e^{-2\pi i \gamma}}{2} = \frac{z + z^{-1}}{2},
\]

\[
\sin^2(\pi \gamma) = \frac{1}{2} (1 - \cos(2\pi \gamma)) = \frac{1}{2} - \frac{z + z^{-1}}{4},
\]

\[
\cos^2(\pi \gamma) = \frac{1}{2} (1 + \cos(2\pi \gamma)) = \frac{1}{2} + \frac{z + z^{-1}}{4}.
\]

3. Now we can define

\[
\cos^{2N}(\pi \gamma) P_{N-1}(\sin^2(\pi \gamma)) = \left( \frac{1}{2} + \frac{z + z^{-1}}{4} \right)^N P_{N-1}\left( \frac{1}{2} - \frac{z + z^{-1}}{4} \right)
\]

\[
= \sum_{m=-2N+1}^{2N-1} a_m z^m \equiv P_{2N-1}(z).
\]

4. So that we are dealing with a true polynomial, we can now define

\[
\tilde{P}_{4N-2}(z) = z^{2N-1} P_{2N-1}(z) = \sum_{m=0}^{4N-2} a_{m+2N-1} z^m
\]

and deal directly with the following factorization problem: Find a polynomial \( B_{2N-1}(z) \) satisfying \( |B_{2N-1}(z)|^2 = \tilde{P}_{4N-2}(z) \). In this case we would have \( m_0(\gamma) = B_{2N-1}(e^{2\pi i \gamma}) \).

**Theorem 0.5** \( \tilde{P}_{4N-2}(z) = |B_{2N-1}(z)|^2 \) where

\[
B_{2N-1}(z) = \text{const.} \cdot (z + 1)^N \prod_{z_0 \in Z_R} (z - z_0) \prod_{z_0 \in Z_C} (z - z_0)(z - \overline{z_0})
\]

where

\[
Z_R = \{ z_0 \in \mathbb{R}: \tilde{P}_{4N-2}(z_0) = 0, |z_0| < 1 \}
\]

and

\[
Z_C = \{ z_0 \in \mathbb{C}: \tilde{P}_{4N-2}(z_0) = 0, |z_0| < 1, \Im(z_0) > 0 \}.
\]

E. Further Examples.

(a) \( N = 2 \).

\[
\tilde{P}_6(z) = \frac{1}{32}(-1 + 9z^2 + 16z^3 + 9z^4 - z^6).
\]

We factor

\[
\tilde{P}_6(z) = \frac{1}{32}(z + 1)^4(-z^2 + 4z - 1) - \frac{1}{32}(z + 1)^4(z - (2 - \sqrt{3}))(z - (2 + \sqrt{3})).
\]
Therefore,

\[ B_3(z) = \frac{1}{4\sqrt{2}} \left( z + 1 \right)^2 \left( z - (2 - \sqrt{3}) \right)^{-1/2} \left( z - (2 - \sqrt{3}) \right) \]
\[ = \frac{1 + \sqrt{3}}{8} \left( z + 1 \right)^2 \left( z - (2 - \sqrt{3}) \right) \]
\[ = \frac{1 + \sqrt{3}}{8} z^3 + \frac{3 + \sqrt{3}}{8} z^2 + \frac{3 - \sqrt{3}}{8} z + \frac{1 - \sqrt{3}}{8}. \]

(b) \( N = 3 \).

\[ \widetilde{\mathcal{P}}_{10}(z) = \frac{1}{512} (3 - 25 z^2 + 75 z^4 + 256 z^5 + 75 z^6 - 25 z^8 + 3 z^{10}). \]

We factor

\[ \widetilde{\mathcal{P}}_{10}(z) = \frac{1}{512} (z + 1)^6 (3z^4 - 18z^3 + 38z^2 - 18z + 3) \]
\[ = \frac{3}{512} (z + 1)^6 (z - \alpha)(z - \overline{\alpha})(z - \alpha^{-1})(z - \overline{\alpha}^{-1}), \]

where \( \alpha \approx .2873 + .1529i \) and

\[ B_5(z) = \frac{\sqrt{3}}{|\alpha|16\sqrt{2}} \left( z + 1 \right)^3 (z - \alpha)(z - \overline{\alpha}) \]