Multiresolution Analysis.

A. Definition and Examples.

Definition 0.1 *A* multiresolution analysis *on* $\mathbf{R}$ *is a sequence of subspaces* $\{V_j\}_{j\in\mathbf{Z}} \subseteq L^2(\mathbf{R})$ *satisfying:*

(a) *For all* $j \in \mathbf{Z}$, $V_j \subseteq V_{j+1}$.

(b) $\overline{\text{span}}\{V_j\}_{j\in\mathbf{Z}} = L^2(\mathbf{R})$. *That is, the set* $\cup_{j\in\mathbf{Z}} V_j$ *is dense in* $L^2(\mathbf{R})$.

(c) $\cap_{j\in\mathbf{Z}} V_j = \{0\}$.

(d) *A function* $f(x) \in V_0$ *if and only if* $D_{2^j} f(x) \in V_j$.

(e) *There exists a function* $\varphi(x)$, L^2 *on* $\mathbf{R}$, *called the* scaling function *such that the collection* $\{T_n\varphi(x)\}$ *is an orthonormal basis for* V_0.

Remarks. (a) An MRA is completely determined by the scaling function $\varphi(x)$ in the following way. Given φ with the property that $\{T_n\varphi(x)\}$ is an orthonormal system, define the subspace V_0 by $V_0 = \overline{\text{span}}\{T_n\varphi(x)\}$, and the subspaces V_j by $V_j = D_{2^j} V_0$, that is, $f \in V_j$ if and only if $D_{2^{-j}} f \in V_0$. Then verify that (a)–(e) hold for this sequence of subspaces.
(b) For example, if we let $\varphi(x) = \chi_{[0,1]}(x)$, then the MRA so generated is called the *Haar MRA* and leads to the construction of the Haar wavelet.
(c) If we let $\varphi(x)$ be defined by $\widehat{\varphi}(\gamma) = \chi_{[-1/2,1/2]}(\gamma)$, then the MRA so generated is called the *bandlimited MRA* and leads to the construction of the Bandlimited wavelet.
(d) If we let $\varphi(x)$ be defined by

$$\widehat{\varphi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \geq 4/3 \\ 1 & \text{if } |\gamma| \leq 1/3 \\ c(\gamma - 1/2) & \text{if } x \in (1/3, 2/3) \\ s(\gamma + 1/2) & \text{if } x \in (-2/3, -1/3) \end{cases}$$

where $s(\gamma)$ and $c(\gamma)$ are as defined in Lecture 3, then the MRA so generated is called the *Meyer MRA* and leads to the construction of the Meyer wavelet.

Lemma 0.1 *Given* $\varphi \in L^2(\mathbf{R})$, *the system* $\{T_n\varphi(x)\}$ *is an orthonormal system if and only if*

$$\sum_{n\in\mathbf{Z}} |\widehat{\varphi}(\gamma + n)|^2 \equiv 1.$$

Corollary 0.1 *Suppose that the system* $\{T_n\varphi(x)\}$ *is an orthonormal system. Then a function* $f \in \overline{\text{span}}\{T_n\varphi(x)\}$ *if and only if there is a sequence* $\{c_n\} \in l^2$ *such that* $f = \sum_n c_n T_n\varphi$ *or equivalently if there is a period-1 function* $C \in L^2[0,1]$ *such that* $\widehat{f}(\gamma) = C(\gamma)\,\widehat{\varphi}(\gamma)$. **(Exercise 1. Verify this.)**

Lemma 0.2 $\{D_{2^j}T_k\varphi\}_{k\in\mathbf{Z}}$ *is an orthonormal basis for* V_j. **(Exercise 2. Verify this.)**

Lemma 0.3 *There exists* $\{h(k)\} \in \ell^2$ *such that* $\varphi(x) = \sum_k h(k)\, 2^{1/2}\varphi(2x-k)$. *This equation is referred to as the* two-scale dilation equation *and the sequence* $\{h(k)\}$ *is referred to as the* scaling sequence *or* scaling filter. *We may write* $\widehat{\varphi}(\gamma) = m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2)$, *where* $m_0(\gamma) = \frac{1}{\sqrt{2}}\sum_k h(k)\, e^{-2\pi i k\gamma}$ *is called the* auxiliary function. **(Exercise 3. Verify the last statement in this Lemma.)**

Lemma 0.4 *If* $\{T_n\varphi(x)\}$ *is an orthonormal system and if* $\varphi(x)$ *satisfies the two-scale dilation equation with scaling filter* $\{h(k)\}$. *Then the auxiliary function* $m_0(\gamma)$ *satisfies*

$$|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2 \equiv 1.$$

Remark. Our goal in this lecture is to prove the following theorem, which will be restated in more detail below: *If* $\{V_j\}$ *is an MRA, then there exists a function* $\psi \in L^2(\mathbf{R})$ *such that* $\{\psi_{j,k}\}$ *is an orthonormal wavelet basis for* $L^2(\mathbf{R})$.

B. Projectors and Subspaces.

Recall that if H is a Hilbert space and M a closed subspace of H, then given $x \in H$, we can decompose x uniquely as $x = x_M + x_{M^\perp}$ where $x_M \in M$ and $x_{M^\perp} \in M^\perp$. The operator P_M that maps x to x_M is called the *orthogonal projector onto* M. The basic result here is the following.

Theorem 0.1 *An operator* P *on a Hilbert space* H *is the orthogonal projector onto* $Ran(P)$ *if and only if* $P^2 = P$ *and* P *is self-adjoint, that is,* $\langle Px, y\rangle = \langle x, Py\rangle$ *for all* x, $y \in H$. *Also, if* P *is an orthogonal projector then for all* $x \in H$, $\|Px\| \le \|x\|$.

Definition 0.2 *For each* $j \in Z$, *define the* approximation operator P_j *to be the orthogonal projector onto* V_j *and the* detail operator Q_j *by* $Q_j = P_{j+1} - P_j$. *Let* W_j *be the orthogonal complement of* V_j *in* V_{j+1}. *In other words,* W_j *consists of all* $f \in V_{j+1}$ *such that* $\langle f, g\rangle = 0$ *for all* $g \in V_j$.

Lemma 0.5 $P_j f = \sum_k \langle f, \varphi_{j,k}\rangle\, \varphi_{j,k}$.

Lemma 0.6 *For all* $f \in L^2(\mathbf{R})$, $P_j f \to f$ *and* $P_{-j} f \to 0$ *as* $j \to \infty$.

Lemma 0.7 Q_j *is the orthogonal projector onto* W_j.

C. Recipe for constructing wavelet bases.

Theorem 0.2 *Let* $\{V_j\}$ *be an MRA with scaling function* $\varphi(x)$ *and scaling filter* $h(k)$. *Define the* wavelet filter $g(k)$ *by* $g(k) = (-1)^k\,\overline{h(1-k)}$ *and the* wavelet $\psi(x)$ *by* $\psi(x) = \sum_k g(k)\, 2^{1/2}\,\varphi(2x-k)$. *Then* $\{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ *is a wavelet orthonormal basis on* $\mathbf{R}$.

Alternatively, given any $J \in \mathbf{Z}$,

$$\{\varphi_{J,k}(x)\}_{k\in\mathbf{Z}} \cup \{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$$

is an orthonormal basis on $\mathbf{R}$.

Remarks. (a) Taking the Fourier transform gives that $\widehat{\psi}(\gamma) = m_1(\gamma/2)\,\widehat{\varphi}(\gamma/2)$, where $m_1(\gamma) = e^{-2\pi i(\gamma+1/2)}\,\overline{m_0(\gamma+1/2)}$. **(Exercise 4. Verify this statement.)**

(b) The alternate form for the wavelet basis is based on the observation that in a traditional wavelet basis, the "low frequency" components of a function f, that is, that portion of $\widehat{f}(\gamma)$ for γ near zero, are the coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j<J,\, k\in\mathbf{Z}}$ where J is any fixed integer. These components can be equivalently and more conveniently expressed in the coefficients $\{\langle f, \varphi_{J,k}\rangle\}_{k\in\mathbf{Z}}$.

The proof of the theorem is accomplished by proving the following lemmas.

Lemma 0.8 *Suppose that $\psi \in L^2$ has the property that $\{T_n\psi\}_{n\in\mathbf{Z}}$ is an orthonormal basis for W_0. Then $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$.*

Claim 1: $\{D_{2^j}T_n\psi\}_{n\in\mathbf{Z}}$ is an orthonormal basis for W_j. **(Exercise 5. Verify this claim.)**
Claim 2: $\operatorname{span}\{D_{2^j}T_n\psi\}$ is dense in $L^2(\mathbf{R})$.

Lemma 0.9 *Suppose that $\psi(x)$ is given by the recipe in the Theorem. Then $\{T_n\psi\}_{n\in\mathbf{Z}}$ is an orthonormal basis for W_0.*

Claim 1: With ψ so defined, the collection $\{T_n\psi\}_{n\in\mathbf{Z}}$ is an orthonormal set. (Exercise 6. Verify this. Hint: Use Lemma 0.1.)
Claim 2: $W_0 = \overline{\operatorname{span}}\{T_n\psi\}_{n\in\mathbf{Z}}$.

Examples. (a) *The Haar wavelet.* In this case, we can compute the scaling and wavelet filters directly.

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) + \frac{1}{\sqrt{2}}\varphi_{1,1}(x).$$

Therefore,

$$h(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0,\ 1, \\ 0 & \text{if } n \neq 0,\ 1, \end{cases}$$

Therefore,

$$g(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, \\ -\frac{1}{\sqrt{2}} & \text{if } n = 1, \\ 0 & \text{if } n \neq 0,\ 1. \end{cases}$$

and

$$\psi(x) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) - \frac{1}{\sqrt{2}}\varphi_{1,1}(x) = \varphi(2x) - \varphi(2x-1) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x).$$

(b) *The Bandlimited wavelet.* Here it is more convenient to work on the transform side. Recall that $\widehat{\varphi}(\gamma) = \chi_{[-1/2,1/2)}(\gamma)$. Since $\widehat{\varphi}(\gamma/2) = \chi_{[-1,1)}(\gamma)$, it follows that

$$\widehat{\varphi}(\gamma) = m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2),$$

where $m_0(\gamma)$ is the period 1 extension of $\chi_{[-1/4,1/4)}(\gamma)$.

Thus, $m_1(\gamma)$ is the period 1 extension of the function

$$e^{-2\pi i(\gamma+1/2)}\Big(\chi_{[-1/2,-1/4)}(\gamma)+\chi_{[1/4,1/2)}(\gamma)\Big)$$

so that

$$\widehat{\psi}(\gamma)=m_1(\gamma/2)\,\widehat{\varphi}(\gamma/2)=-e^{-\pi i\gamma}\Big(\chi_{[-1,-1/2)}(\gamma)+\chi_{[1/2,1)}(\gamma)\Big).$$

By taking the inverse Fourier transform,

$$\psi(x)=\frac{\sin(2\pi x)-\cos(\pi x)}{\pi(x-1/2)}=\frac{\sin\pi(x-1/2)}{\pi(x-1/2)}\,(1-2\sin\pi x).$$

(c) *The Meyer wavelet.* Recall that

$$\widehat{\varphi}(\gamma)=\begin{cases}0 & \text{if } |\gamma|\geq 2/3,\\ 1 & \text{if } |\gamma|\leq 1/3,\\ s(\gamma+1/2) & \text{if } \gamma\in(1/3,2/3),\\ c(\gamma-1/2) & \text{if } \gamma\in(-2/3,-1/3),\end{cases}$$

Therefore, $\widehat{\varphi}(\gamma)=m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2)$, where $m_0(\gamma)$ is the period 1 extension of the function

$$\widehat{\varphi}(2\gamma)\,\chi_{[-1/2,1/2]}(\gamma).$$

$\psi(x)$ is defined by

$$\widehat{\psi}(\gamma)=-e^{-\pi i\gamma}\,\overline{m_0(\gamma/2+1/2)}\,\widehat{\varphi}(\gamma/2)$$

and

$$\widehat{\psi}(\gamma)=\begin{cases}0 & \text{if } |\gamma|\leq 1/3 \text{ or } |\gamma|\geq 4/3,\\ s(\gamma-1/2) & \text{if } \gamma\in(1/3,2/3],\\ c(\gamma/2-1/2) & \text{if } \gamma\in(2/3,4/3),\\ s(\gamma/2+1/2) & \text{if } \gamma\in(-4/3,-2/3),\\ c(\gamma+1/2) & \text{if } \gamma\in[-2/3,-1/3).\end{cases}$$