Orthonormal Bases in Hilbert Space.

Linear (Vector) Spaces.

**Definition 0.1** A linear space is a nonempty set $L$ together with a mapping from $L \times L$ into $L$ called addition, denoted $(x, y) \mapsto x + y$ and a mapping from the Cartesian product of either $\mathbb{R}$ or $\mathbb{C}$ with $L$ into $L$ called scalar multiplication, denoted $(\alpha, x) \mapsto \alpha x$, which satisfy the following properties.

(1) Axioms of addition.

(a) $x + y = y + x$ (commutativity).

(b) $(x + y) + z = x + (y + z)$ (associativity).

(c) There exists an element $0 \in L$, called the zero element, such that for all $x \in L$, $0 + x = x + 0 = x$.

(d) For every $x \in L$, there is an element $-x \in L$ such that $x + (-x) = (-x) + x = 0$.

(2) Axioms of scalar multiplication.

(a) $\alpha(\beta x) = (\alpha \beta)x$, for all scalars $\alpha, \beta$ and all $x \in L$.

(b) $1(x) = x$.

(3) Distributive Laws.

(a) $(\alpha + \beta)x = \alpha x + \beta x$, for all scalars $\alpha, \beta$ and all $x \in L$.

(b) $\alpha(x + y) = \alpha x + \alpha y$, for all scalars $\alpha$ and all $x, y \in L$.

Examples. (a) $L = \mathbb{R}^n$ or $\mathbb{C}^n$ with addition and scalar multiplication defined component-wise in the usual way.

(b) $L = C[0, 1]$, the space of functions continuous on $[0, 1]$, with addition of functions defined by $(f + g)(x) = f(x) + g(x)$, and scalar multiplication defined by $(\alpha f)(x) = \alpha f(x)$.

c) The following spaces are examples of sequence spaces and are in some sense the natural generalizations of $\mathbb{R}^n$ as $n \to \infty$.

$$\ell^\infty = \{\{x_n\}_{n=1}^\infty : \sup |x_n| < \infty\}$$

$$c_0 = \{\{x_n\}_{n=1}^\infty : \lim x_n = 0\}$$

$$\ell^2 = \{\{x_n\}_{n=1}^\infty : \sum |x_n|^2 < \infty\}$$

$$\ell^1 = \{\{x_n\}_{n=1}^\infty : \sum |x_n| < \infty\}$$

$$f = \{\{x_n\}_{n=1}^\infty : \exists N = N(x), x_n = 0 \forall n \geq N\},$$

where $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\alpha \{x_n\} = \{\alpha x_n\}$.

**Exercise 1.** Show that $f \subset \ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty$ as sets.
Definition 0.2 A subset $L'$ of a linear space $L$ is a subspace of $L$ provided that for every $x, y \in L'$ and scalars $\alpha, \beta$,
$$\alpha x + \beta y \in L'.$$
In other words, $L'$ is a subspace provided that it is a linear space in its own right with addition and scalar multiplication inherited from $L$.

Given a nonempty subset $S$ of a linear space $L$, the linear span or span of $S$, denoted $\text{span}(S)$, is the set of all finite linear combinations of elements in $S$, that is
$$\text{span}(S) = \{\sum_{i=1}^{n} \alpha_i x_i : \alpha_i \text{ scalars}, x_i \in S\}.$$

Proposition 0.1 Given a nonempty subset $S$ of a linear space $L$, $\text{span}(S)$ is a subspace of $L$ and moreover, $\text{span}(S)$ is the smallest subspace of $L$ containing $S$, that is, if $L'$ is any subspace containing $S$, then $\text{span}(S) \subseteq L'$.

Proof: Exercise 2.

Normed Linear Spaces.

Definition 0.3 A normed linear space is a pair $(V, \| \cdot \|)$ where $V$ is a linear space (over $\mathbb{R}$ or $\mathbb{C}$), and $\| \cdot \|$ is a function $\| \cdot \| : V \rightarrow \mathbb{R}$ called a norm which satisfies for all $v, w \in V$ the following properties.

1. $\|v\| \geq 0$.
2. $\|v\| = 0$ if and only if $v = 0$.
3. $\|\alpha v\| = |\alpha|\|v\|$ for every scalar $\alpha$.
4. $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality).

Remark 0.1 Given a normed linear space $V$, we can define a metric on $V$ by $d(x, y) = \|x - y\|$, making $(V, d)$ a metric space.

Examples.

1. $(C[0, 1], \| \cdot \|_{\infty})$, where $\|f\|_{\infty} = \sup_{x \in [0, 1]} |f(x)|$.
2. $(C[0, 1], \| \cdot \|_{1})$, where $\|f\|_1 = \int_0^1 |f(x)| \, dx$.
3. $(C[0, 1], \| \cdot \|_{2})$, where $\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx\right)^{1/2}$.
4. $(\mathbb{R}^n, \| \cdot \|_2)$, where $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$.
5. $(\mathbb{R}^n, \| \cdot \|_1)$, where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. 


6. \((\mathbb{R}^n, \| \cdot \|_\infty)\), where \(\|x\|_\infty = \max_{1=1,...,n} |x_i|\).

7. \(\ell^1\) where \(\|\{x_n\}\|_1 = \sum_n |x_n|\).

8. \(\ell^2\) where \(\|\{x_n\}\|_2 = (\sum_n |x_n|^2)^{1/2}\).

9. \(\ell^\infty\) where \(\|\{x_n\}\|_\infty = \sup_n |x_n|\).

**Exercise 3.** Show that for all \(f \in C[0,1]\),

(a) \(\|f\|_1 \leq \|f\|_\infty\)

(b) \(\|f\|_2 \leq \|f\|_\infty\)

(c) \(\|f\|_1 \leq \|f\|_2\) (Use the Cauchy-Schwarz inequality proved later.)

Show that in each case an opposite inequality does not hold in the sense that (for example) there is no number \(C > 0\) such that for all \(f \in C[0,1]\), \(\|f\|_\infty \leq C \|f\|_1\). In other words, for each number \(C\), find a function \(f \in C[0,1]\) (which will depend on \(C\)) such that \(\|f\|_\infty > C \|f\|_1\). Find corresponding examples for (b) and (c).

**Exercise 4.** Show that for all \(x \in C^n\),

(a) \(\|x\|_\infty \leq \|x\|_1\)

(b) \(\|x\|_\infty \leq \|x\|_2\)

(c) \(\|x\|_2 \leq \|x\|_1\)

(d) \(\|x\|_2 \leq \sqrt{n} \|x\|_\infty\)

(e) \(\|x\|_1 \leq n \|x\|_\infty\).

Use these inequalities to show that the three norms are equivalent.

**Exercise 5.** Show that for all complex-valued sequences \(x = \{x_n\}_{n=1}^\infty\)

(a) \(\|x\|_\infty \leq \|x\|_1\)

(b) \(\|x\|_\infty \leq \|x\|_2\)

(c) \(\|x\|_2 \leq \|x\|_1\).

Show that in each case an opposite inequality does not hold in the same sense as in Exercise 3. That is, show that (for example) given \(C > 0\) there is a sequence \(x = \{x_n\}\) (which will depend on \(C\)) such that \(\|x\|_1 > C \|x\|_\infty\).
Inner Product and Hilbert Spaces.

Definition 0.4 An inner product space is a pair $(V, ⟨·, ·⟩)$ where $V$ is a vector space over $\mathbb{C}$ or $\mathbb{R}$ and where $⟨·, ·⟩$ is a complex valued function

$$⟨·, ·⟩: V \times V \rightarrow \mathbb{C}$$

called the inner product on $V$ satisfying the following properties. For all $x, y, z \in V$ and $α \in \mathbb{C}$,

1. $⟨x, x⟩ \geq 0$ and $⟨x, x⟩ = 0$ if and only if $x = 0$.
2. $⟨αx, y⟩ = α⟨x, y⟩$.
3. $⟨x + z, y⟩ = ⟨x, y⟩ + ⟨z, y⟩$.
4. $⟨x, y⟩ = ⟨y, x⟩$.

Examples.

1. $\mathbb{R}^n$ with $⟨x, y⟩ = \sum_{i=1}^{n} x_i y_i$.
2. $V = C[0, 1]$ with $⟨f, g⟩ = \int_0^1 f(x)g(x) \, dx$.
3. $V = ℓ^2$ with $⟨x, y⟩ = \sum_{n=1}^{\infty} x_n \overline{y_n}$.

Theorem 0.1 Let $V$ be an inner product space. Define the function

$$∥·∥: V \rightarrow \mathbb{R}$$

by $∥x∥ = ⟨x, x⟩^{1/2}$. Then $∥·∥$ is a norm on $V$ making $V$ a normed linear space.

Corollary 0.1 (Cauchy-Schwarz Inequality). For $x, y \in V$, an inner product space,

$$|⟨x, y⟩| \leq ∥x∥∥y∥.$$ 

Exercise 6. Prove the Cauchy-Schwarz inequality for a real inner product space. (Hint: For $t \in \mathbb{R}$, and $x, y \in V$, note that $0 \leq ∥x + ty∥^2 = ⟨x + ty, x + ty⟩$ defines a real-valued, non-negative quadratic function of $t$. What must such a quadratic satisfy?)

Exercise 7. Prove the triangle inequality for the norm defined through the inner product as in the Theorem. (Hint: Expand $∥x + y∥^2$ as an inner product and use the C-S inequality).

Definition 0.5 A Hilbert space is a complete inner product space. (Complete here means that every Cauchy sequence converges.)
**Examples.**

1. \( L^2[0,1] \) with the inner product given by 
   \[
   \langle f, g \rangle = \int_0^1 f(x) g(x) \, dx
   \]
   is a Hilbert space.

2. \( \ell^2 \) with the inner product given by 
   \[
   \langle \{x_n\}, \{y_n\} \rangle = \sum_n x_n \overline{y_n}
   \]
   is a Hilbert space.

3. \( L^2(\mathbb{R}) \) with the inner product given by 
   \[
   \langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) \, dx
   \]
   is a Hilbert space.

**Exercise 8.** Show that the sequence space \( f \) of finite sequences is not complete by finding a sequence in \( f \) which is Cauchy but which does not converge to anything in \( f \).

**Orthonormal systems in Hilbert spaces.**

**Definition 0.6** Let \( V \) be an inner product space. Two vectors \( x, y \in V \) are said to be **orthogonal** if \( \langle x, y \rangle = 0 \). We also write in this case \( x \perp y \).

A collection of vectors \( \{x_\alpha\}_{\alpha \in A} \subseteq V \) is said to be an **orthonormal system** if \( \langle x_\alpha, x_\beta \rangle = 0 \) for \( \alpha \neq \beta \) and if \( \langle x_\alpha, x_\alpha \rangle = 1 \) for all \( \alpha \in A \).

**Lemma 0.1** (Best Approximation Lemma). Let \( \{x_n\}_{n=1}^N \) be an orthonormal system in an inner product space \( V \) and let \( \{a_n\}_{n=1}^N \) be a finite sequence of scalars. Then for all \( x \in V \),
\[
\|x - \sum_{n=1}^N a_n x_n\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n \rangle|^2 + \sum_{n=1}^N |a_n - \langle x, x_n \rangle|^2.
\]

**Corollary 0.2**

1. \( \sum_{n=1}^N |a_n|^2 = \sum_{n=1}^N |a_n|^2 \).

2. \( \|x - \sum_{n=1}^N \langle x, x_n \rangle x_n\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n \rangle|^2 \).

3. (Bessel’s Inequality) \( \sum_{n=1}^N |\langle x, x_n \rangle|^2 \leq \|x\|^2 \).

4. If \( \{x_n\}_{n=1}^\infty \) is an orthonormal system, then for each \( x \in V \), the series \( \sum_{n=1}^\infty |\langle x, x_n \rangle|^2 \)
   converges and \( \sum_{n=1}^\infty |\langle x, x_n \rangle|^2 \leq \|x\|^2 \).
Orthonormal bases in Hilbert spaces.

Definition 0.7 A collection of vectors \( \{ x_\alpha \}_{\alpha \in A} \) in a Hilbert space \( H \) is complete if \( \langle y, x_\alpha \rangle = 0 \) for all \( \alpha \in A \) implies that \( y = 0 \).

An equivalent definition of completeness is the following. \( \{ x_\alpha \}_{\alpha \in A} \) is complete in \( V \) if \( \text{span}\{ x_\alpha \} \) is dense in \( V \), that is, given \( y \in H \) and \( \epsilon > 0 \), there exists \( y' \in \text{span}\{ x_\alpha \} \) such that \( \| x - y' \| < \epsilon \). Another way to put this is that given \( y \), every ball around \( y \) contains an element of \( \text{span}\{ x_\alpha \} \). The proof of this equivalence relies on a fundamental decomposition property of Hilbert spaces.

An orthonormal basis a complete orthonormal system.

Theorem 0.2 Let \( \{ x_n \}_{n=1}^\infty \) be an orthonormal system in a Hilbert space \( H \). Then the following are equivalent.

1. \( \{ x_n \} \) is complete.
2. For all \( x \in H \),
   \[ x = \sum_{n=1}^\infty \langle x, x_n \rangle x_n, \]
   where the sum converges unconditionally, that is, regardless of order.
3. \( \| x \|^2 = \sum_{n=1}^\infty |\langle x, x_n \rangle|^2 \).
4. For all \( x, y \in H \),
   \[ \langle x, y \rangle = \sum_{n=1}^\infty \langle x, x_n \rangle \overline{\langle y, x_n \rangle}. \]

Examples.

1. \( \mathbb{R}^n \) or \( \mathbb{C}^n \). In finite dimensional vector spaces we have the notion of linear independence and dimension. Specifically if the finite dimensional vector space \( X \) has dimension \( N \) and if \( V = \{ v_k \}_{k=1}^N \) is an orthonormal system, then it is an orthonormal basis. Any collection of \( N \) linearly independent vectors can be orthogonalized via the Gram-Schmidt process into an orthonormal basis.

2. \( L^2[0, 1] \) is the space of all Lebesgue measurable functions on \([0, 1]\), square-integrable in the sense of Lebesgue. This space can be thought of as the completion of the incomplete normed linear space \( C[0, 1] \) with respect to the norm \( \| f \|_2^2 = \int_0^1 |f(x)|^2 \, dx \). In other words, each element of \( L^2[0, 1] \) can be thought of as the limit (in the sense of the \( L^2 \) norm) of a Cauchy sequence of continuous functions.

The trigonometric system \( \{ e^{2\pi i nx} \}_{n=-\infty}^\infty \) is an orthonormal basis for \( L^2[0, 1] \). The expansion of a function in this basis is called the Fourier series of that function.

Another example of an orthonormal basis for \( L^2[0, 1] \) are the Legendre polynomials which are obtained by taking the sequence of monomials \( \{ 1, x, x^2, \ldots \} \) and applying the Gram-Schmidt orthogonalization process to it.
3. $L^2(\mathbb{R})$ is the space of all Lebesgue measurable functions on $\mathbb{R}$, square-integrable in the sense of Lebesgue. This space can be thought of as the completion of the incomplete normed linear space $C_c(\mathbb{R})$ of functions continuous on $\mathbb{R}$ with compact support (equipped with the $L^2$ norm), or as the completion of the incomplete normed linear space $C_c^\infty(\mathbb{R})$ of all compactly supported, infinitely differentiable functions on $\mathbb{R}$ equipped again with the $L^2$ norm. This will be our setting for much of our discussion of wavelet bases. We will show how to construct orthonormal bases of this space with wavelets.

**Exercise 9.** Show that the trigonometric system $\{e^{2\pi inx}\}_{n=-\infty}^{\infty}$ is an orthonormal system in $L^2[0, 1]$.

**Exercise 10.** Find the first four Legendre polynomials by applying the Gram-Schmidt process to the sequence $\{1, x, x^2, \ldots\}$. 