## Lecture 9: Banach Spaces.

**Definition 71.** *A normed linear space is* **complete** *if it is complete as a metric space in the metric induced by the norm. A complete normed linear space is called a* **Banach space***.*

**Theorem 72.** *Let* $T\colon (V_1, \|\cdot\|_1) \longrightarrow (V_2, \|\cdot\|_2)$ *be a bounded linear transformation and suppose that* $(V_2, \|\cdot\|_2)$ *is complete. Then* $T$ *can be extended uniquely to a bounded linear transformation* $\widetilde{T}$ *from the completion of* $V_1$*,* $\widetilde{V_1}$*, to* $V_2$ *in such a way that* $\|T\| = \|\widetilde{T}\|$*. (*$\widetilde{T}$ *extends* $T$ *means that for all* $v \in V_1$*,* $T(v) = \widetilde{T}(v)$*).*

**Proof:** Let $\widetilde{V_1}$ be the completion of $V_1$.

Claim 1. $\widetilde{V_1}$ is a normed linear space with the obvious definition of addition and scalar multiplication, and with the norm defined by $\|v\|_{\widetilde{V_1}} = \lim \|v_n\|_1$ where $\{v_n\}$ is a Cauchy sequence in $V_1$ converging to $v \in \widetilde{V_1}$. We will use $\|\cdot\|_1$ also to denote the norm on $\widetilde{V_1}$.

We must define $\widetilde{T}$. Let $x \in \widetilde{V_1}$, and let $\{x_n\}$ be a Cauchy sequence in $V_1$ converging to $x$. We must show that the sequence $T(x_n)$ is Cauchy in $V_2$. To see this, let $\epsilon > 0$. Then there is an $N$ so that if $n$, $m \geq N$ then $\|x_n - x_m\|_1 < \epsilon/\|T\|$. Then if $n$, $m \geq N$,

$$\|T(x_n) - T(x_m)\|_2 = \|T(x_n - x_m)\|_2 \leq \|T\|\|x_n - x_m\|_1 < \epsilon.$$

Therefore, $\{T(x_n)\}$ is Cauchy in $V_2$ and hence converges in $V_2$. Define $\widetilde{T}(x) = \lim T(x_n)$.

Claim 2. The definition of $\widetilde{T}$ is independent of the choice of the approximating sequence $\{x_n\}$.

Proof of Claim 2. Let $x \in \widetilde{V_1}$, and suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $V_1$ converging to $x$. It is easy to see that the sequence $\{x_n - y_n\}$ is Cauchy in $V_1$ and converges to $0 \in V_1$. Since $T$ is bounded, $T(x_n - y_n) \to 0$ in $V_2$. Now suppose that $z_1 = \lim T(x_n)$ and $z_2 = \lim T(y_n)$. Then

$$\|z_1 - z_2\|_2 \leq \|z_1 - T(x_n)\|_2 + \|T(x_n) - T(y_n)\|_2 + \|T(y_n) - z_2\|_2.$$

Since each term on the right side of the inequality can be made as small as desired, $\|z_1 - z_2\|_2 = 0$ or $z_1 = z_2$.

Claim 3. $\widetilde{T}$ is linear.

Claim 4. $\widetilde{T}$ is bounded with $\|\widetilde{T}\| = \|T\|$.

Proof of Claim 4. Let $x \in \widetilde{V_1}$ and let $\{x_n\}$ be a Cauchy sequence in $V_1$ converging to $x$. Then

$$\begin{aligned} \|\widetilde{T}(x)\|_2 &= \lim \|T(x_n)\|_2 \\ &\leq \lim \|T\|\|x_n\|_1 = \|T\|\|x\|_1. \end{aligned}$$

This shows also that $\|\widetilde{T}\| \leq \|T\|$. To see that in fact equality holds, note that

$$\|\widetilde{T}\| = \sup_{x\in\widetilde{V}_1} \frac{\|\widetilde{T}(x)\|_2}{\|x\|_1} \geq \sup_{x\in V_1} \frac{\|\widetilde{T}(x)\|_2}{\|x\|_1} = \sup_{x\in V_1} \frac{\|T(x)\|_2}{\|x\|_1} = \|T\|.$$

Thus, $\|\widetilde{T}\| \geq \|T\|$, so $\|\widetilde{T}\| = \|T\|$.

Claim 5. The extension $\widetilde{T}$ is unique.
Proof of Claim 5. Suppose that there were two extensions $\widetilde{T}_1$ and $\widetilde{T}_2$. Let $x \in \widetilde{V}_1$ and let $\{x_n\}$ be a Cauchy sequence in $V_1$ converging to $x$. Since $\widetilde{T}_1$ and $\widetilde{T}_2$ are both extensions of $T$,

$$\widetilde{T}_1(x_n) = T(x_n) = \widetilde{T}_2(x_n),$$

and since limits in metric spaces are unique,

$$\widetilde{T}_1(x) = \lim T(x_n) = \widetilde{T}_2(x).\square$$

The notion of the extension of an operator from a dense subset of a space to the whole space is very important in concrete situations. Often an operator can be defined easily only on a dense subspace of the space you are interested in. If you can get the proper bound on the operator then it extends to a continuous operator in the whole space, i.e.,

**Corollary 73.** *Let* $(V_1, \|\cdot\|_1)$, $(V_2, \|\cdot\|_2)$ *be normed linear spaces with* $V_2$ *complete. Let* $B \subseteq V_1$ *be a dense subspace and suppose that there is a linear operator defined on* $B$ *such that there is a constant* $C > 0$ *such that for all* $v \in B$, $\|Tv\|_2 \leq C\|v\|_1$. *Then* $T$ *can be extended uniquely to a bounded linear operator from the completion of* $V_1$ *to* $V_2$.

**Proof:** The proof follows from Theorem 72 and the observation that the completion of $(B, \|\cdot\|_1)$ is the same as the completion of $(V_1, \|\cdot\|_1)$. $\square$

**Example 74.** (a) Let $C[0,1]$ be equipped with the norm

$$\|f\|_1 = \int_0^1 |f(x)|\,dx.$$

We have seen that this space is not complete. The space $L^1[0,1]$ is defined to be the completion of $C[0,1]$ in this norm. The question remains: What to objects in $L^1$ look like? We have defined them as Cauchy sequences. Is there any sense in which these objects are actually functions with point values? We will see the answer to this when we discuss the Lebesgue measure and Lebesgue integral.

(b) The completion of the open interval $(0,1)$ in $\mathbf{R}$ is the closed interval $[0,1]$. This is simply the original set with all limit points added.

(c) The space $L^2[0,1]$ which can be defined as the completion of $C[0,1]$ under the norm $\|\cdot\|_2$ defined by

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}.$$

(d) The space $C[0,1]$ with the norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

(e) The space $\ell^\infty$ defined by

$$\ell^\infty = \{x = \{x_n\}_{n=1}^\infty \subset \mathbf{C}\colon \|x\|_\infty = \sup_{n\in\mathbf{N}} |x_n| < \infty\}.$$

(f) The space $\ell^p$, $1 \le p < \infty$ defined by

$$\ell^p = \{x = \{x_n\}_{n=1}^\infty \subset \mathbf{C}\colon \|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} < \infty\}.$$

(g) The space $c_0$ defined by

$$c_0 = \{x = \{x_n\}_{n=1}^\infty \subset \mathbf{C}\colon \lim_{n\to\infty} x_n = 0\},$$

equipped with the same norm as $\ell_\infty$.

(h) The space $f$ defined by

$$f = \{x = \{x_n\}_{n=1}^\infty \subset \mathbf{C}\colon \exists N = N(x),\ x_n = 0\ \forall n \ge N\},$$

equipped with the same norm as $\ell_\infty$, is not complete. In fact, $f$ is dense in $\ell^p$, $1 \le p \le \infty$ and also in $c_0$.