

Lecture 2: Topology of Metric Spaces

Definition 13. Let (\mathcal{X}, d) be a metric space and let $\mathcal{M} \subseteq \mathcal{X}$. Then the pair (\mathcal{M}, d) is a metric space in its own right and is called a **subspace** of (\mathcal{X}, d) .

Definition 14. Let (\mathcal{X}, d) be a metric space. The open ball of radius $r > 0$ about $x \in \mathcal{X}$ is the set $B(x; r) = \{y: d(x, y) < r\}$. The closed ball is the set $B(x; r) = \{y: d(x, y) \leq r\}$.

A subset $\mathcal{U} \subseteq \mathcal{X}$ is **open** if for each $x \in \mathcal{U}$, there is an $\epsilon > 0$ such that $B(x; \epsilon) \subseteq \mathcal{U}$. A set $\mathcal{F} \subseteq \mathcal{X}$ is **closed** if its complement \mathcal{F}^c is open.

Given $x \in \mathcal{X}$, a set $\mathcal{N} \subseteq \mathcal{X}$ is a **neighborhood** of x if there is an open ball $B(x; r) \subseteq \mathcal{N}$.

Theorem 15. (Theorem 3, p. 49, Theorem 5, p. 50, Kolmogorov)

(a) Let $\{\mathcal{U}_\alpha\}$ be a system (finite, countable or uncountable) of open sets in a metric space X . Then $\cup_\alpha \mathcal{U}_\alpha$ is open.

(b) Let $\{\mathcal{U}_n\}_{n=1}^N$ be a finite system of open sets in \mathcal{X} . Then $\cap_{n=1}^N \mathcal{U}_n$ is open.

Proof: (a) Let $x \in \cup_\alpha \mathcal{U}_\alpha$, then for some fixed α_0 , $x \in \mathcal{U}_{\alpha_0}$. Since \mathcal{U}_{α_0} is open, there exists $\epsilon > 0$ such that $B(x; \epsilon) \subseteq \mathcal{U}_{\alpha_0}$. Thus also, $B(x; \epsilon) \subseteq \cup_\alpha \mathcal{U}_\alpha$ and $\cup_\alpha \mathcal{U}_\alpha$ is open.

(b) Let $x \in \cap_{n=1}^N \mathcal{U}_n$. Then $x \in \mathcal{U}_n$ for all n and for each n there is an $\epsilon_n > 0$ such that $B(x; \epsilon_n) \subseteq \mathcal{U}_n$. Let ϵ be the smallest of the ϵ_n . Then $B(x; \epsilon) \subseteq B(x; \epsilon_n) \subseteq \mathcal{U}_n$ for all n and hence $B(x; \epsilon) \subseteq \cap_{n=1}^N \mathcal{U}_n$. Therefore $\cap_{n=1}^N \mathcal{U}_n$ is open. \square

Corollary 16.

(a) Let $\{\mathcal{F}_\alpha\}$ be a system (finite, countable or uncountable) of closed sets in a metric space X . Then $\cap_\alpha \mathcal{F}_\alpha$ is closed.

(b) Let $\{\mathcal{F}_n\}_{n=1}^N$ be a finite system of closed sets in \mathcal{X} . Then $\cup_{n=1}^N \mathcal{F}_n$ is closed.

Example 17. (a) The set $(0, 1) \subseteq \mathbf{R}$ is open in \mathbf{R} with the usual absolute value metric. The set $[0, 1]$ is closed in \mathbf{R} because its complement, the union of two open intervals, is an open set. The sets \emptyset and \mathbf{R} are both open and closed as subsets of \mathbf{R} . The set $(0, 1]$ is neither open nor closed in \mathbf{R} .

(b) Any singleton (that is, a set consisting of only one point) in \mathbf{R} is closed. The set $[0, 1] \cup \{2\}$ is closed in \mathbf{R} as it is the union of two closed sets.

(c) Any subset \mathcal{M} of a metric space (\mathcal{X}, d) where d is the discrete topology is closed.

(d) Let $\mathcal{F}_n = [1/n, 1 - 1/n]$. Then each \mathcal{F}_n is closed but $\cup_{n=1}^\infty \mathcal{F}_n = (0, 1)$ which is not closed.

(e) Let $\mathcal{U}_n = (-1/n, 1 + 1/n)$. Then each \mathcal{U}_n is open but $\bigcap_{n=1}^{\infty} \mathcal{U}_n = [0, 1]$ which is not open.

Definition 18. *The point $x \in \mathcal{X}$ is a closure point of $\mathcal{M} \subseteq \mathcal{X}$ if for every $\epsilon > 0$ $B(x; \epsilon)$ contains a point of \mathcal{M} . (Kolmogorov calls such a point a contact point.)*

The point $x \in \mathcal{X}$ is a limit point or an accumulation point of \mathcal{M} if for every $\epsilon > 0$, $B(x; \epsilon)$ contains infinitely many points of \mathcal{M} . Equivalently, $x \in \mathcal{X}$ is a limit point of \mathcal{M} if for all $\epsilon > 0$, $B(x; \epsilon)$ contains a point of $\mathcal{M} \setminus \{x\}$.

A closure point of \mathcal{M} which is not a limit point of \mathcal{M} is called an isolated point.

Lemma 19.

- (a) *A point x is a closure point of \mathcal{M} if and only if there is a sequence of points $\{x_n\}$ of \mathcal{M} converging to x .*
- (b) *A point x is a limit point of \mathcal{M} if and only if there is a sequence of distinct points $\{x_n\}$ of \mathcal{M} converging to x .*
- (c) *An isolated point of \mathcal{M} must be in \mathcal{M} .*

Proof: (a) For each n , let $\epsilon = 1/n$ and choose $x_n \in B(x; 1/n) \cap \mathcal{M}$. Then $x_n \in \mathcal{M}$ for each n and $x_n \rightarrow x$.

(b) Use the same procedure as in (a) to choose the x_n but since x is an accumulation point, we can guarantee that x_n is not equal to x nor to any of the (finitely many) previously chosen points.

(c) Suppose that x is an isolated point of \mathcal{M} which is not in \mathcal{M} . Let $\epsilon_1 = 1$ and let x_1 be any point in $B(x; \epsilon_1) \cap \mathcal{M}$. Note that $x_1 \neq x$. Let $\epsilon_2 = \min\{1/2, d(x, x_1)\}$ and choose x_2 to be any point in $B(x; \epsilon_2) \cap \mathcal{M}$. Continue in this fashion, letting $\epsilon_n = \min\{1/n, d(x, x_{n-1})\}$, and choosing x_n to be any point in $B(x; \epsilon_n) \cap \mathcal{M}$. Since $\epsilon_n \leq 1/n$, $x_n \rightarrow x$, and since $d(x, x_n) < d(x, x_{n-1})$, x_n is not equal to any of the previously chosen x_m . Thus $\{x_n\}$ is a sequence of distinct points in \mathcal{M} converging to x . By definition then x is a limit point of \mathcal{M} and hence cannot be an isolated point. \square

Definition 20. *The closure of \mathcal{M} is the set of all closure points of \mathcal{M} , and is denoted $[\mathcal{M}]$. Note that the closure of \mathcal{M} consists of \mathcal{M} together with all of its limit points. A subspace (\mathcal{M}, d) is **closed** provided that \mathcal{M} is closed.*

Example 21. (a) The set $\mathcal{M} = (0, 1) \subseteq \mathbf{R}$ is not closed in \mathbf{R} because 1 is a closure point of \mathcal{M} but is not in \mathcal{M} .

(b) The point 2 is a closure point of the set $\mathcal{M} = [0, 1] \cup \{2\}$. In fact it is an isolated point of \mathcal{M} .

(c) Any subset \mathcal{M} of a metric space (\mathcal{X}, d) where d is the discrete topology is closed.

Lemma 22. *Let \mathcal{M} be a subspace of a metric space \mathcal{X} . Then the following are equivalent.*

- (a) \mathcal{M} is closed.
- (b) $[\mathcal{M}] = \mathcal{M}$.
- (c) Every convergent sequence $\{x_n\} \subseteq \mathcal{X}$ satisfying $\{x_n\} \subseteq \mathcal{M}$ converges to an element of \mathcal{M} .

Proof: (a) \iff (b). Assume that \mathcal{M} is closed. Since $[\mathcal{M}]$ consists of \mathcal{M} together with its limit points, always $\mathcal{M} \subseteq [\mathcal{M}]$. Suppose $x \notin \mathcal{M}$. Since \mathcal{M} is closed its complement is open and since x is in that complement there is an $\epsilon > 0$ such that $B(x; \epsilon) \cap \mathcal{M} = \emptyset$. But by definition, this means that x is not a closure point of \mathcal{M} so that $x \notin [\mathcal{M}]$.

Now assume that $[\mathcal{M}] = \mathcal{M}$. It will be sufficient to show that $[\mathcal{M}]$ is closed. If $x \notin [\mathcal{M}]$ then x is not a closure point of \mathcal{M} so that there is an $\epsilon > 0$ such that $B(x; \epsilon) \cap \mathcal{M} = \emptyset$. But since $\mathcal{M} = [\mathcal{M}]$, $B(x; \epsilon) \cap [\mathcal{M}] = \emptyset$ also. Therefore the complement of $[\mathcal{M}]$ is open and $[\mathcal{M}]$ is closed.

(b) \iff (c). Assume that $[\mathcal{M}] = \mathcal{M}$ and that and let $\{x_n\}$ be a sequence in \mathcal{M} converging to the point x . By the definition of convergence, for every $\epsilon > 0$ there is at least one of the x_n in $B(x; \epsilon)$. Hence x is a closure point of \mathcal{M} and so is in $[\mathcal{M}]$. But since $[\mathcal{M}] = \mathcal{M}$, $x \in \mathcal{M}$.

Now suppose that (b) does not hold and that there is a point $x \in [\mathcal{M}]$ which is not in \mathcal{M} . But by Lemma 19(a) there is a sequence $\{x_n\}$ in \mathcal{M} converging to x . But this means that (c) also does not hold. \square

Theorem 23. *(Properties of Closure.)*

- (a) If $\mathcal{M} \subseteq \mathcal{N}$ then $[\mathcal{M}] \subseteq [\mathcal{N}]$.
- (b) $[[\mathcal{M}]] = [\mathcal{M}]$.
- (c) $[\mathcal{M} \cup \mathcal{N}] = [\mathcal{M}] \cup [\mathcal{N}]$.

Proof: (a) Let $x \in [\mathcal{M}]$. Then for every $\epsilon > 0$, $B(x; \epsilon) \cap \mathcal{M} \neq \emptyset$. Since $\mathcal{M} \subseteq \mathcal{N}$, $B(x; \epsilon) \cap \mathcal{N} \neq \emptyset$ also. Hence $x \in [\mathcal{N}]$.

(b) This follows immediately from Lemma 22 and the fact that $[\mathcal{M}]$ is closed.

(c) By part (a), and since both \mathcal{M} and \mathcal{N} are subsets of $\mathcal{M} \cup \mathcal{N}$, $[\mathcal{M}] \cup [\mathcal{N}] \subseteq [\mathcal{M} \cup \mathcal{N}]$. By definition of closure, $\mathcal{M} \subseteq [\mathcal{M}]$ and $\mathcal{N} \subseteq [\mathcal{N}]$, so that $\mathcal{M} \cup \mathcal{N} \subseteq [\mathcal{M}] \cup [\mathcal{N}]$. By part (a), $[\mathcal{M} \cup \mathcal{N}] \subseteq [[\mathcal{M}] \cup [\mathcal{N}]] = [\mathcal{M}] \cup [\mathcal{N}]$ by Lemma 22 and the fact that the union of two closed sets is closed. \square

Lemma 24. *A closed subspace of a complete metric space is complete.*

Proof: Let \mathcal{M} be a subspace of the complete metric space \mathcal{X} , and let $\{x_n\}$ be a Cauchy sequence in \mathcal{M} . Then also, $\{x_n\}$ is a Cauchy sequence in \mathcal{X} and hence converges to some $x \in \mathcal{X}$. Therefore, x is a closure point of \mathcal{M} but since \mathcal{M} is closed, $x \in \mathcal{M}$. Hence \mathcal{M} is complete. \square