Signals and Systems.

**Definition.** A *signal* is a sequence of numbers $\{x(n)\}_{n\in\mathbf{Z}}$ satisfying $\sum_{n\in\mathbf{Z}}|x(n)| < \infty$. Such a sequence is also referred to as being in $\ell^1(\mathbf{Z})$, or just in $\ell^1$. A sequence $\{x(n)\}$ satisfying $\sum_{n\in\mathbf{Z}}|x(n)|^2 < \infty$ is referred to as an $\ell^2$ sequence.

**Definition.** The *frequency domain representation* of a signal $x(n)$ is the function

$$\widehat{x}(\omega) = \sum_{n\in\mathbf{Z}} x(n)\, e^{-2\pi i n\omega} = X(e^{2\pi i\omega}).$$

We think of the function $X(e^{2\pi i\omega})$ as the restriction to the unit circle in the complex plane of some function $X(z)$ defined on some portion of the complex plane containing the unit circle. In this case, $X(z)$ is defined as

$$X(z) = \sum_{n\in\mathbf{Z}} x(n)\, z^{-n}$$

and is referred to as the $z$*–transform* of $x(n)$.

**Definition.** **(a)** A *system* is any transformation $T$ that takes an input signal $x(n)$ to an output signal $y(n)$. We write $Tx(n) = y(n)$.
**(b)** A system $T$ is *linear* if

$$T(a\,x_1 + b\,x_2)(n) = a\,Tx_1(n) + b\,Tx_2(n)$$

where $x_1, x_2 \in \ell^1$, and $a, b$ are constants.
**(c)** A linear system $T$ is *stable* if for some $C > 0$

$$\sum_{n \in \mathbf{Z}} |Tx(n)| \le C \sum_{n \in \mathbf{Z}} |x(n)|$$

for all signals $x(n)$.
**(d)** For $n_0 \in \mathbf{Z}$, the *translation operator* $\tau_{n_0}$, for signals is $\tau_{n_0} x(n) = x(n - n_0)$.
**(e)** A *linear translation–invariant (LTI) system* is a linear system $T$ for which

$$T(\tau_{n_0} x)(n) = \tau_{n_0}(Tx)(n) = Tx(n - n_0).$$

**(f)** The *convolution* of signals $x_1, x_2 \in \ell^1$, denoted $x_1 * x_2(n)$, is $y(n) = x_1 * x_2(n) = \sum_{n \in \mathbf{Z}} x_1(k)\, x_2(n-k)$.

**Theorem.** **(a)** If $x_1, x_2 \in \ell^1$, then $y = x_1 * x_2 \in \ell^1$.
**(b)** If $x_1, x_2 \in \ell^1$, $x_1 * x_2 = x_2 * x_1$.
**(c)** Let $h \in \ell^1$, and define the transformation $T_h$ by $T_h x(n) = (x * h)(n)$. Then $T_h$ is a stable LTI system.

**Theorem.** Let $T$ be a stable LTI system. Then there is an $h \in \ell^1$ such that

$$Tx(n) = (x * h)(n) = \sum_{k \in \mathbf{Z}} x(k)\, h(n-k).$$

The signal $h$ is called the *impulse response* of $T$. The impulse response of a stable LTI system is often called a *filter*. The frequency representation of $h(n)$, $\widehat{h}(\omega)$, is called the *frequency response* of $T$, and the $z$–transform of $h(n)$, $H(z)$, is called the *system function* of $T$.

**Theorem.** Let $x_1, x_2 \in \ell^1$, and let $y = x_1 * x_2$. Then

$$\widehat{y}(\omega) = \widehat{x}_1(\omega)\, \widehat{x}_2(\omega).$$

**Definition.** Given $N \in \mathbf{N}$, a sequence $\{x(n)\}_{n\in\mathbf{Z}}$ is a *period $N$ signal* if $x(n+N) = x(n)$ for all $n \in \mathbf{Z}$.

**Theorem.** Given a filter $h(n)$ and a period $N$ signal $x(n)$, the convolution $x * h(n)$ is defined for all $n$ and is a period $N$ signal.

**Definition.** Given a period $N$ signal $x(n)$, the ($N$–*point*) *Discrete Fourier Transform* or ($N$–*point*) *DFT* of $x(n)$, denoted $\widehat{x}(n)$, is the period $N$ sequence defined by

$$\widehat{x}(n) = \sum_{j=0}^{N-1} x(j)\, e^{-2\pi i j n/N}.$$

**Theorem.** Given a period $N$ sequence $x(n)$ with DFT $\widehat{x}(n)$,

$$x(j) = \frac{1}{N} \sum_{n=0}^{N-1} \widehat{x}(n)\, e^{2\pi i n j/N},$$

for each $j \in \mathbf{Z}$.

**Theorem.** Let $h(n)$ be a filter and $x(n)$ a period $N$ signal. Then

$$(x * h)^{\wedge}(n) = \widehat{x}(n)\, \widehat{h}(n/N),$$

where $\widehat{x}(n)$ is the DFT of $x(n)$ and $\widehat{h}(\omega)$ is the frequency representation of $h$.

**Definition.** Let $x(n)$ and $y(n)$ be period-$N$ signals. Then the *circular convolution* of $x(n)$ and $y(n)$ is defined by

$$x * y(n) = \sum_{k=0}^{N-1} x(k)\, y(n-k).$$

**Remark.** Circular convolution can be realized as multiplication by a matrix whose rows are shifts of one another. Given a period $N$ sequence $x(n)$, define the matrix $X$ by

$$X = \begin{pmatrix} x(0) & x(N-1) & x(N-2) & \cdots & x(1) \\ x(1) & x(0) & x(N-1) & \cdots & x(2) \\ & & \cdots & & \\ x(N-1) & x(N-2) & x(N-3) & \cdots & x(0) \end{pmatrix}.$$

If $y$ has period $N$ and $r(n) = x{*}y(n)$, define $\mathbf{y} = [y(0) \ \cdots \ y(N-1)]$ and $\mathbf{r} = [r(0) \ \cdots \ r(N-1)]$. Then $\mathbf{r} = X\mathbf{y}$.

**Theorem.** Let $x(n)$ and $y(n)$ be period-$N$ signals, and let $\widehat{x}(n)$ and $\widehat{y}(n)$ be their DFTs; then

$$(x * y)^{\wedge}(n) = \widehat{x}(n)\,\widehat{y}(n),$$

where $(x * y)^{\wedge}(n)$ denotes the DFT of $x * y(n)$.

# The DFT in MATLAB

The command `fft` computes the DFT in MATLAB.

```
FFT Discrete Fourier transform.
    FFT(X) is the discrete Fourier transform (DFT) of vector X.  For
    matrices, the FFT operation is applied to each column. For N-D
    arrays, the FFT operation operates on the first non-singleton
    dimension.

    FFT(X,N) is the N-point FFT, padded with zeros if X has less
    than N points and truncated if it has more.

    FFT(X,[],DIM) or FFT(X,N,DIM) applies the FFT operation across the
    dimension DIM.

    For length N input vector x, the DFT is a length N vector X,
    with elements
                      N
       X(k) =       sum  x(n)*exp(-j*2*pi*(k-1)*(n-1)/N), 1 <= k <= N.
                     n=1
    The inverse DFT (computed by IFFT) is given by
                      N
       x(n) = (1/N) sum  X(k)*exp( j*2*pi*(k-1)*(n-1)/N), 1 <= n <= N.
                     k=1

    See also IFFT, FFT2, IFFT2, FFTSHIFT.
```

A quick comparison of the DFT and the DHT.

Under MATLAB's definition of the DFT, we have the following notion of symmetry for the DFT.

**Theorem.** **(a)** If $x(n)$ has period $N$ and is real-valued, then $X(k) = \overline{X(2-k)}$.
**(b)** If $x(n)$ is real-valued and satisfies $x(n) = x(2-n)$, then so does $X(k)$.

**Remark.** The DHT of a real-valued signal of length $N$ ($N = 2^n$) consists of $N$ real numbers, the Haar coefficients. The DFT of the same signal consists of $N$ complex numbers. By symmetry, this can be reduced to $N$ real numbers, the real numbers $X(1)$ and $X((N/2)-1)$, and the real and imaginary parts of the numbers strictly between $k=1$ and $k=(N/2)-1$. A similar thing holds when $N$ is odd.

## Some small MATLAB examples.

```
>> x=[1 1 1 1 1 1 1 1];
>> y=fft(x)
y =
     8     0     0     0     0     0     0     0
>> x1=[1 2 3 4 5 4 3 2]
x1 =
     1     2     3     4     5     4     3     2
>> y1=fft(x1)
y1 =
   24.0000   -6.8284  0  -1.1716   0 -1.1716  0  -6.8284
>> x1=[1 2 3 4 5 6 7 8];
>> y1=fft(x1)
y1 =
  Columns 1 through 6
  36.0000  (-4.0000 + 9.6569i)  (-4.0000 + 4.0000i)
  (-4.0000 + 1.6569i) -4.0000  (-4.0000 - 1.6569i)
  Columns 7 through 8
  (-4.0000 - 4.0000i)  (-4.0000 - 9.6569i)
>> x2=[1 2 3 4 5 6 7];
>> y2=fft(x2)
y2 =
  Columns 1 through 6
  28.0000  (-3.5000 + 7.2678i) (-3.5000 + 2.7912i)
  (-3.5000 + 0.7989i) (-3.5000 - 0.7989i) (-3.5000 - 2.7912i)
  Column 7
  (-3.5000 - 7.2678i)
```

```
>> x=[1 1 1 1 1 0 0 0];
>> y=fft(x)
y =
  Columns 1 through 6
   5.0000   (0 - 2.4142i)   1.0000
   (0 - 0.4142i) 1.0000  (0 + 0.4142i)
  Columns 7 through 8
   1.0000   (0 + 2.4142i)
>> yy=wavedec(x,4,'haar')
yy =
    2.5000  0  1.0607  0  0.5000  0  0  0.7071  0
>> x1=[1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0];
>> y1=fft(x1)
y1 =
  Columns 1 through 6
  11.0000  (-1.6310 - 3.9375i) (1.7071 - 1.7071i)
  (0.3244 + 0.1344i)  (0 - 1.0000i) (1.0898 - 0.4514i)
  Columns 7 through 12
   (0.2929 + 0.2929i)  (0.2168 - 0.5233i)   1.0000
   (0.2168 + 0.5233i)  (0.2929 - 0.2929i)   (1.0898 + 0.4514i)
  Columns 13 through 16
   (0 + 1.0000i)  (0.3244 - 0.1344i) (1.7071 + 1.7071i)
   (-1.6310 + 3.9375i)
>> yy1=wavedec(x1,4,'haar')
yy1 =
  Columns 1 through 10
    2.7500    1.2500   0  1.0607    0   0   0.5000   0   0   0
  Columns 11 through 16
    0    0    0    0.7071    0    0
```

The DHT is noticeably sparser in the latter case.