The Fourier transform.

Definition. The *Fourier transform* of a function $f \in L^1(\mathbf{R})$, is also a function on $\mathbf{R}$, denoted $\widehat{f}(\gamma)$ defined by

$$\widehat{f}(\gamma) = \int_{\mathbf{R}} f(x)\, e^{-2\pi i\gamma x}\, dx.$$

Theorem. If $f(x)$ is L^1 on $\mathbf{R}$, then $\widehat{f}(\gamma)$ is uniformly continuous on $\mathbf{R}$.

Proof:

Theorem. (Riemann-Lebesgue Lemma) If $f(x)$ is L^1 on $\mathbf{R}$, then

$$\lim_{|\gamma|\to\infty} \widehat{f}(\gamma) = 0.$$

Theorem. (Fourier Inversion) If $f \in L^1(\mathbf{R})$, and if $\widehat{f} \in L^1(\mathbf{R})$, then for each $x \in \mathbf{R}$,

$$\int_{\mathbf{R}} \widehat{f}(\gamma)\, e^{2\pi i\gamma x}\, d\gamma = f(x).$$

Convolution.

Definition. Given functions $f(x)$ and $g(x)$, the *convolution* of $f(x)$ and $g(x)$, denoted $h(x) = f * g(x)$, is defined by

$$f * g(x) = \int_{\mathbf{R}} f(t)\, g(x - t)\, dt.$$

Remarks. (a) The convolution $f * g(x)$ can be interpreted as a "moving weighted average" of $f(x)$, where the "weighting" is determined by the function $g(x)$.

(b) Convolution is *commutative*, i.e., $f * g(x) = g * f(x)$.

(c) Convolution is a *smoothing operation*. In general, $f * g(x)$ will be *at least as smooth* as the smoothest of f and g, and if both f and g are smooth, it will pick up the smoothness of both.

(d) The convolution of f and g will in general decay at infinity *only as fast as the slowest-decaying of f or g.*

Theorem. If $f \in L^\infty(\mathbf{R})$ and $g \in L^1(\mathbf{R})$, or if $f, g \in L^2(\mathbf{R})$, then $f * g \in C^0(\mathbf{R})$.

Theorem. **(a)** If $f, g \in L^1(\mathbf{R})$, then $f * g \in L^1(\mathbf{R})$, and

$$\|f * g\|_1 \le \|f\|_1 \, \|g\|_1.$$

(b) If $f \in L^1(\mathbf{R})$, and $g \in L^2(\mathbf{R})$, then $f * g \in L^2(\mathbf{R})$, and

$$\|f * g\|_2 \le \|f\|_1 \, \|g\|_2.$$

(c) If $f, g \in L^2(\mathbf{R})$, then $f * g \in L^\infty(\mathbf{R})$, and

$$\|f * g\|_\infty \le \|f\|_2 \, \|g\|_2.$$

(d) If $f \in L^\infty(\mathbf{R})$, and $g \in L^1(\mathbf{R})$, then $f * g \in L^\infty(\mathbf{R})$, and

$$\|f * g\|_\infty \le \|f\|_\infty \, \|g\|_1.$$

Theorem. (The Convolution Theorem) If $f, g \in L^1(\mathbf{R})$, then

$$\widehat{f * g}(\gamma) = \widehat{f}(\gamma) \, \widehat{g}(\gamma).$$

Plancherel's and Parseval's formula.

Theorem. (Plancherel's Formula) If $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\widehat{f} \in L^2(\mathbf{R})$ and

$$\int_{\mathbf{R}} |\widehat{f}(\gamma)|^2 \, d\gamma = \int_{\mathbf{R}} |f(x)|^2 \, dx.$$

Theorem. (Parseval's Formula) If $f,\, g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, then

$$\int_{\mathbf{R}} \widehat{f}(\gamma) \, \overline{\widehat{g}(\gamma)} \, d\gamma = \int_{\mathbf{R}} f(x) \, \overline{g(x)} \, dx.$$

Smoothness and Decay.

Remark. A fundamental principle of use in interpreting many results about the Fourier transform is the following: *Smooth functions have Fourier transforms that decay rapidly to zero at infinity, and functions that decay rapidly to zero at infinity have smooth Fourier transforms.*

Theorem. (Differentiation Theorem) If f, $x\,f(x) \in L^1(\mathbf{R})$, then $\widehat{f} \in C^1(\mathbf{R})$, and

$$\widehat{xf}(\gamma) = \frac{-1}{2\pi i}\,\frac{d\widehat{f}}{d\gamma}(\gamma).$$

Corollary. If f, $x^N f(x) \in L^1(\mathbf{R})$ for some $N \in \mathbf{N}$, then $\widehat{f} \in C^N(\mathbf{R})$, and for $0 \le j \le N$,

$$\widehat{x^j f}(\gamma) = (\frac{-1}{2\pi i})^j\,\frac{d^j\widehat{f}}{d\gamma^j}(\gamma).$$

Corollary. If f, $\widehat{f}$, $\gamma^N \widehat{f}(\gamma) \in L^1(\mathbf{R})$, then $f \in C^N(\mathbf{R})$, and for $0 \le j \le N$,

$$f^{(j)}(x) = \int_{\mathbf{R}} (2\pi i\gamma)^j\,\widehat{f}(\gamma)\,e^{2\pi i\gamma x}\,d\gamma.$$

Theorem. Suppose that $f \in L^1(\mathbf{R})$, and that for some $N \in \mathbf{N}$,

(1) $\widehat{f} \in C^N(\mathbf{R})$,

(2) $\widehat{f}$, $\widehat{f}^{(N)} \in L^1(\mathbf{R})$

(3) For $0 \le j \le N$, $\lim\limits_{|\gamma|\to\infty} \widehat{f}^{(j)}(\gamma) = 0$. Then

$$\lim_{|x|\to\infty} x^N f(x) = 0.$$

Dilation, Translation, and Modulation.

Recall the following definition, to which we add one more.

Definition Given $a > 0$, the *dilation operator*, D_a is given by

$$D_a f(x) = a^{1/2} f(ax).$$

Given $b \in \mathbf{R}$, the *translation operator*, T_b is given by

$$T_b f(x) = f(x - b).$$

Given $c \in \mathbf{R}$, the *modulation operator*, E_c is given by

$$E_c f(x) = e^{2\pi i c x} f(x).$$

Theorem. Let $f \in L^1(\mathbf{R})$.
(a) For every $a > 0$, $\widehat{D_a f}(\gamma) = D_{1/a}\widehat{f}(\gamma)$.
(b) For every $b \in \mathbf{R}$, $\widehat{T_b f}(\gamma) = E_{-b}\widehat{f}(\gamma)$.
(c) For every $c \in \mathbf{R}$, $\widehat{E_c f}(\gamma) = T_c\widehat{f}(\gamma)$.

Theorem. ((Further) Properties of Dilation and Translation) For every f, $g \in L^2(\mathbf{R})$, and for every $a > 0$, $b \in \mathbf{R}$,

(c) $\langle f, D_a g\rangle = \langle D_{a^{-1}} f, g\rangle$.

(d) $\langle f, T_b g\rangle = \langle T_{-b} f, g\rangle$.

(e) $\langle f, D_a T_b g\rangle = \langle T_{-b} D_{a^{-1}} f, g\rangle$.

(f) $\langle D_a f, D_a g\rangle = \langle f, g\rangle$.

(g) $\langle T_b f, T_b g\rangle = \langle f, g\rangle$.

Theorem. (Properties of Translation and Modulation) For every f, $g \in L^2(\mathbf{R})$, and for every b, $c \in \mathbf{R}$,

(a) $T_b E_c f(x) = e^{-2\pi i b c}\, E_c T_b f(x)$.

(b) $\langle f, E_c g\rangle = \langle E_{-c} f, g\rangle$.

(c) $\langle f, T_b E_c g\rangle = e^{2\pi i b c}\, \langle T_{-b} E_{-c} f, g\rangle$.

Bandlimited Functions and the Sampling Formula

Definition. A function $f \in L^2(\mathbf{R})$, is *bandlimited* if there is a number $\Omega > 0$ such that $\widehat{f}(\gamma)$ is supported in the interval $[-\Omega/2, \Omega/2]$. In this case, the function $f(x)$ is said to have *bandlimit* $\Omega > 0$.

Theorem. (The Shannon Sampling Theorem) If $f(x)$ is bandlimited with bandlimit Ω, then $f(x)$ can be written as

$$f(x) = \sum_n f(n/\Omega)\,\frac{\sin(\pi\Omega(x - n/\Omega))}{\pi\Omega(x - n/\Omega)},$$

in L^2 and L^∞ on $\mathbf{R}$.