Fourier series.

Definition. A function $f(x)$ defined on $\mathbf{R}$ *has period* $p > 0$ if $f(x+p) = f(x)$ for all $x \in \mathbf{R}$. Such a function is said to be *(p) periodic*.

Given $a > 0$, the collection of functions

$$\{e^{2\pi i n x/a}\}_{n\in\mathbf{Z}} = \{e_{n/a}(x)\}_{n\in\mathbf{Z}}$$

is called the *(period a) trigonometric system*. Here $e_\lambda(x) = e^{2\pi i \lambda x}$.

Theorem. The period a trigonometric system is an orthogonal system on $[0, a]$ (and in fact on any interval I of length a), that is,

$$\begin{aligned} \langle e_{n/a}, e_{m/a}\rangle &= \int_0^a e^{2\pi i n x/a}\, e^{-2\pi i m x/a}\, dx \\ &= \begin{cases} 0 & \text{if } n \neq m, \\ a & \text{if } n = m. \end{cases} \end{aligned}$$

The fundamental problem in Fourier series is the following: Given a function $f(x)$ with period $a > 0$, we want to write

$$f(x) = \sum_{n \in \mathbf{Z}} c(n)\, e^{2\pi i n x/a}$$

This leads to three questions:

(a) (**Coefficients**) What are the coefficients $c(n)$?

(b) (**Convergence**) In what sense does the series converge?

(c) (**Completeness or Uniqueness**) Does the series on the right converge to $f(x)$, or to some other function?

Definition. Given a function $f(x)$ with period a, the *Fourier coefficients* of $f(x)$ are defined for each $n \in \mathbf{Z}$ by

$$c(n) = \frac{1}{a}\,\langle f, e_{n/a}\rangle = \frac{1}{a}\int_0^a f(x)\, e^{-2\pi i n x/a}\, dx.$$

The answers to questions (b) and (c) occupied mathematicians for many decades. Here are a couple of famous theorems.

Theorem. (Dirichlet) Suppose that $f(x)$ has period $a > 0$ and is piecewise differentiable on $\mathbf{R}$. Then the sequence of partial sums of the Fourier series of $f(x)$, $\{S_N(x)\}_{N \in \mathbf{N}}$, where

$$S_N(x) = \sum_{n=-N}^{N} c(n)\, e^{2\pi i n x/a},$$

converges pointwise to the function $\widetilde{f}(x)$, where

$$\widetilde{f}(x) = \frac{1}{2}\left[\lim_{t \to x^+} f(t) + \lim_{t \to x^-} f(t)\right].$$

Theorem. (Fejér) Let $f(x)$ be a function with period $a > 0$ continuous on $\mathbf{R}$, and define for each $n \in \mathbf{N}$ the function $\sigma_N(x)$ by

$$\sigma_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} S_k(x).$$

Then $\sigma_N \to f$ in L^∞ on $\mathbf{R}$.

Generalized Fourier Series.

Definition. Recall that a collection of functions $\{g_n(x)\}_{n\in\mathbf{N}}$ on I is an *orthogonal system* (on I) provided that
(a) $\langle g_n, g_m\rangle = 0$ if $n \neq m$, and
(b) $\langle g_n, g_n\rangle > 0$,
and an *orthonormal system* on I if in addition
(b') $\langle g_n, g_n\rangle = 1$.

Definition. Given a function $f \in L^2(I)$, and an orthonormal system $\{g_n(x)\}$ on I, the *(generalized) Fourier coefficients*, $\{c(n)\}$ of $f(x)$ with respect to $\{g_n(x)\}$ are defined by

$$c(n) = \int_I f(x)\,\overline{g(x)}\,dx = \langle f, g_n\rangle.$$

We want to write $f(x) = \sum_{n\in\mathbf{N}} \langle f, g_n\rangle\, g_n(x)$. The questions of convergence and completeness remain.

Theorem. (Bessel's Inequality) Let $f \in L^2(I)$, and let $\{g_n(x)\}$ be an orthonormal system on I. Then

$$\sum_{n \in \mathbf{N}} |\langle f, g_n \rangle|^2 \leq \|f\|_2^2.$$

Lemma. Let $\{g_n(x)\}$ be an orthonormal system on I. Then for every $f \in L^2(I)$, and every $N \in \mathbf{N}$,

$$\|f - \sum_{n=1}^{N} \langle f, g_n \rangle \, g_n\|_2^2 = \|f\|_2^2 - \sum_{n=1}^{N} |\langle f, g_n \rangle|^2.$$

Lemma. Let $\{g_n(x)\}$ be an orthonormal system on I. Then for every $f \in L^2(I)$, and every finite sequence of numbers $\{a(n)\}_{n=1}^{N}$,

$$\begin{aligned} \|f - \sum_{n=1}^{N} a(n) \, g_n\|_2^2 \; &= \; \|f - \sum_{n=1}^{N} \langle f, g_n \rangle \, g_n\|_2^2 \\ &\quad + \sum_{n=1}^{N} |a(n) - \langle f, g_n \rangle|^2. \end{aligned}$$

Definition. Given a collection of functions $\{g_n\} \subseteq L^2(I)$, the *span of* $\{g_n\}$, denoted span$\{g_n\}$, is the collection of all finite linear combinations of the elements of $\{g_n\}$. In other words, $f \in$ span$\{g_n\}$ if and only if $f(x) = \sum_{n=1}^{N} a(n)\, g_n(x)$ for some finite sequence $\{a(n)\}_{n=1}^{N}$. Note that N is always finite but may be arbitrarily large.

Remark. The definition of span involves only finite sums. In general an infinite series of the form $\sum_{n\in\mathbf{N}} a(n)\, g_n(x)$ will not converge in any sense.

Definition. The L^2 *closure of* span$\{g_n\}$, denoted $\overline{\text{span}}\{g_n\}$ is defined as follows. A function $f(x) \in \overline{\text{span}}\{g_n\}$ if there is a sequence $\{f_n\} \subseteq\in \overline{\text{span}}\{g_n\}$ such that $f_n \to f$. This is different from saying that $f = \sum_{n=1}^{N} a(n)\, g_n$ for some coefficients $\{a(n)\}$ g_n.

Theorem. Let $\{g_n(x)\}$ be an ONS on I. Then $f \in \overline{\text{span}}\{g_n\}$ if and only if

$$f(x) = \sum_{n\in\mathbf{N}} \langle f, g_n\rangle\, g_n(x),$$

in L^2 on I.

Definition. Let $\{g_n(x)\}$ be an orthonormal system on I. Then $\{g_n\}$ is *complete* on I provided that every function $f \in L^2(I)$, is in $\overline{\text{span}}\{g_n\}$. A complete orthonormal system is called an *orthonormal basis*.

Theorem. Let $\{g_n(x)\}$ be an orthonormal system on I. Then the following are equivalent.

(a) $\{g_n\}$ is complete on I.
(b) For every $f \in L^2(I)$,

$$f(x) = \sum_{n \in \mathbf{N}} \langle f, g_n \rangle \, g_n(x)$$

in L^2 on I.
(c) Every function $f \in C_c^0(I)$, is in $\overline{\text{span}}\{g_n(x)\}$.
(d) For every function $f \in C_c^0(I)$,

$$\|f\|_2^2 = \int_I |f(x)|^2 \, dx = \sum_{n \in \mathbf{N}} |\langle f, g_n \rangle|^2.$$

Theorem. The trigonometric system $\{e_{n/a}(x)\}_{n \in \mathbf{Z}}$ is complete on $[0, 1]$.