Multiresolution Analysis.

A new look at the Haar system.

Definition. For each $j \in \mathbf{Z}$, define the *approximation operator* P_j on $L^2(\mathbf{R})$, by

$$P_j f(x) = \sum_k \langle f, p_{j,k} \rangle \, p_{j,k}(x).$$

Define the *approximation space* V_j by

$$V_j = \overline{\text{span}}\{p_{j,k}(x)\}_{k \in \mathbf{Z}}.$$

Since $\{p_{j,k}(x) \colon k \in \mathbf{Z}\}$ is an orthonormal system on $\mathbf{R}$, $P_j f(x)$ is the function in V_j best approximating $f(x)$ in the L^2 sense.

Define the *detail operator* Q_j on $L^2(\mathbf{R})$, by

$$Q_j f(x) = P_{j+1} f(x) - P_j f(x).$$

Define the *wavelet space* W_j by

$$W_j = \overline{\text{span}}\{h_{j,k}(x)\}_{k \in \mathbf{Z}}.$$

Since $\{h_{j,k}(x)\}_{k \in \mathbf{Z}}$ is an orthonormal system on $\mathbf{R}$ $Q_j f(x)$ is the function in W_j best approximating $f(x)$ in the L^2 sense.

Theorem.(a) The scale J Haar system on $\mathbf{R}$ is a complete orthonormal system on $\mathbf{R}$. (The scale J Haar system is

$$\{p_{J,k}(x),\, h_{j,k}(x) \colon j \geq J;\; k \in \mathbf{Z}\}.$$

(b) The Haar system is a complete orthonormal system on $\mathbf{R}$. (The Haar system is

$$\{\, h_{j,k}(x) \colon j\, k \in \mathbf{Z}\}).$$

Proving that the Haar system is a complete orthonormal system on $\mathbf{R}$ amounts to showing the following.

Theorem. (a) $\lim_{j \to \infty} \|P_j f - f\|_2 = 0$, and
(b) $\lim_{j \to \infty} \|P_j f\|_2 = 0$.
(c) Given $f \in C_c^0(\mathbf{R})$,

$$Q_j f(x) = \sum_k \langle f, h_{j,k} \rangle\, h_{j,k}(x).$$

Definition. A *multiresolution analysis* on $\mathbf{R}$ is a sequence of subspaces $\{V_j\}_{j\in\mathbf{Z}} \subseteq L^2(\mathbf{R})$ satisfying:

(a) For all $j \in \mathbf{Z}$, $V_j \subseteq V_{j+1}$.

(b) $\overline{\text{span}}\{V_j\}_{j\in\mathbf{Z}} = L^2(\mathbf{R})$. That is, given $f \in L^2(\mathbf{R})$ and $\epsilon > 0$, there is a $j \in \mathbf{Z}$ and a function $g(x) \in V_j$ such that $\|f - g\|_2 < \epsilon$.

(c) $\cap_{j\in\mathbf{Z}} V_j = \{0\}$.

(d) A function $f(x) \in V_0$ if and only if $D_{2^j} f(x) \in V_j$.

(e) There exists a function $\varphi(x)$, L^2 on $\mathbf{R}$, called the *scaling function* such that the collection $\{T_n\varphi(x)\}$ is an orthonormal system of translates and

$$V_0 = \overline{\text{span}}\{T_n\varphi(x)\}.$$

Examples of MRA.

Note: In order to define an MRA it is sufficient to either (1) specify V_0 then show that there is a scaling function $\varphi(x)$ such that $V_0 = \overline{\text{span}}\{T_n\varphi\}$, or (2) specify the scaling function $\varphi(x)$ and define $V_0 = \overline{\text{span}}\{T_n\varphi\}$.

(a) The Haar MRA. $\varphi(x) = p_{0,0}(x) = \mathbf{1}_{[0,1]}(x)$.

(b) The Bandlimited MRA. V_0 is the set of all functions f bandlimited to $[-1/2, 1/2]$.

(c) The Meyer MRA.

Given $k \in \mathbf{N}$ (or $k = \infty$), a function $b(x)$ is a C^k *bell function over* $[-1/2, 1/2]$ provided that $b(x)$ is C^k on $\mathbf{R}$ and satisfies the following conditions:

(a) $b(x) = 1$ if $|x| \leq 1/3$,

(b) $b(x) = 0$ if $|x| > 2/3$,

(c) $0 \leq b(x) \leq 1$ for all $x \in \mathbf{R}$, and

(d) $\sum_n |b(x+n)|^2 \equiv 1$.

Now take $\varphi(x)$ to be the inverse Fourier transform of a C^k bell-function.

(d) The Piecewise Linear MRA. Let V_0 consist of all functions $f \in L^2(\mathbf{R}) \cap C^0(\mathbf{R})$ linear on the intervals $I_{0,k}$, for $k \in \mathbf{Z}$. Think of this as a stepped-up version of the Haar MRA.

Define the function $\varphi(x) = (1 - |x|)\,\mathbf{1}_{[-1,1]}(x)$.

Lemma. If $f \in V_0$ then $f(x) = \sum_n f(n)\,T_n\varphi(x)$ pointwise *and in* $L^2(\mathbf{R})$.

Lemma. $V_0 = \overline{\text{span}} T_n\varphi$.

Theorem. There is a function $\widetilde{\varphi}(x)$, L^2 on $\mathbf{R}$, such that:
(a) $\{T_n\widetilde{\varphi}(x)\}$ is an orthonormal system of translates, and
(b) $V_0 = \overline{\text{span}}\{T_n\widetilde{\varphi}(x)\}$.

Some results about collections of the form $\{T_n g\}_{n\in\mathbf{Z}}$.

(a) If $\{T_n g\}_{n\in\mathbf{Z}}$ is an orthonormal system on $\mathbf{R}$, then $f \in \overline{\text{span}} T_n g$ if and only if

$$f(x) = \sum_n \langle f, T_n g\rangle\, T_n g(x)$$

in L^2 if and only if there is a Fourier series $\widehat{c}(\gamma)$ with period 1 such that

$$\widehat{f}(\gamma) = \widehat{g}(\gamma)\,\widehat{c}(\gamma).$$

(b) The collection $\{T_n g(x)\}$ is an orthonormal system of translates if and only if for all $\gamma \in \mathbf{R}$,

$$\sum_n |\widehat{g}(\gamma+n)|^2 \equiv 1.$$

(c) If for some $0 < A < B$

$$A \le \sum_n |\widehat{g}(\gamma+n)|^2 \le B$$

then there is a function $\widetilde{g} \in L^2(\mathbf{R})$, such that:

(i) $\{T_n \widetilde{g}(x)\}$ is an orthonormal system of translates and

(ii) $\overline{\text{span}}\{T_n g(x)\} = \overline{\text{span}}\{T_n \widetilde{g}(x)\}$.

Wavelet basis from MRA

Theorem. (The two-scale relation) There exists $\{h(k)\} \in \ell^2$ such that

$$\varphi(x) = \sum_k h(k)\, 2^{1/2} \varphi(2x - k)$$

in L^2 on $\mathbf{R}$. Moreover, we may write

$$\widehat{\varphi}(\gamma) = m_0(\gamma/2)\, \widehat{\varphi}(\gamma/2),$$

where

$$m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k)\, e^{-2\pi i k \gamma}.$$

Theorem. (The wavelet "recipe") Let $\{V_j\}$ be an MRA with scaling function $\varphi(x)$ and scaling filter $h(k)$. Define the *wavelet filter* $g(k)$ by

$$g(k) = (-1)^k \, \overline{h(1-k)}$$

and the *wavelet* $\psi(x)$ by

$$\psi(x) = \sum_k g(k) \, 2^{1/2} \, \varphi(2x - k).$$

Then

$$\{\psi_{j,k}(x)\}_{j,k \in \mathbf{Z}}$$

is a wavelet orthonormal basis on $\mathbf{R}$.

Alternatively, given any $J \in \mathbf{Z}$,

$$\{\varphi_{J,k}(x)\}_{k \in \mathbf{Z}} \cup \{\psi_{j,k}(x)\}_{j,k \in \mathbf{Z}}$$

is an orthonormal basis on $\mathbf{R}$.

Remark. Taking the Fourier transform gives that

$$\widehat{\psi}(\gamma) = m_1(\gamma/2) \, \widehat{\varphi}(\gamma/2),$$

where

$$m_1(\gamma) = e^{-2\pi i(\gamma + 1/2)} \, \overline{m_0(\gamma + 1/2)},$$

(a) The Haar wavelet. In this case, we can compute the scaling and wavelet filters directly.

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) + \frac{1}{\sqrt{2}}\varphi_{1,1}(x).$$

Therefore,

$$h(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0,\ 1, \\ 0 & \text{if } n \neq 0,\ 1, \end{cases}$$

Therefore,

$$g(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, \\ -\frac{1}{\sqrt{2}} & \text{if } n = 1, \\ 0 & \text{if } n \neq 0,\ 1. \end{cases}$$

and

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2}}\varphi_{1,0}(x) - \frac{1}{\sqrt{2}}\varphi_{1,1}(x) \\ &= \varphi(2x) - \varphi(2x-1) \\ &= \mathbf{1}_{[0,1/2)}(x) - \mathbf{1}_{[1/2,1)}(x). \end{aligned}$$

(b) The Bandlimited wavelet. Here it is more convenient to work on the transform side. Recall that $\widehat{\varphi}(\gamma) = \mathbf{1}_{[-1/2,1/2)}(\gamma)$. Since $\widehat{\varphi}(\gamma/2) = \mathbf{1}_{[-1,1)}(\gamma)$, it follows that

$$\widehat{\varphi}(\gamma) = m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2),$$

where $m_0(\gamma)$ is the period 1 extension of $\mathbf{1}_{[-1/4,1/4)}(\gamma)$.

Thus, $m_1(\gamma)$ is the period 1 extension of the function

$$e^{-2\pi i(\gamma+1/2)}\,(\mathbf{1}_{[-1/2,-1/4)}(\gamma) + \mathbf{1}_{[1/4,1/2)}(\gamma))$$

so that

$$\begin{aligned}\widehat{\psi}(\gamma) &= m_1(\gamma/2)\,\widehat{\varphi}(\gamma/2)\\ &= -e^{-\pi i\gamma}\,(\mathbf{1}_{[-1,-1/2)}(\gamma) + \mathbf{1}_{[1/2,1)}(\gamma)).\end{aligned}$$

By taking the inverse Fourier transform,

$$\begin{aligned}\psi(x) &= \frac{\sin(2\pi x) - \cos(\pi x)}{\pi(x-1/2)}\\ &= \frac{\sin\pi(x-1/2)}{\pi(x-1/2)}\,(1 - 2\sin\pi x).\end{aligned}$$

(c) The Meyer wavelet. Recall that

$$\widehat{\varphi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \geq 2/3, \\ 1 & \text{if } |\gamma| \leq 1/3, \\ s(\gamma + 1/2) & \text{if } \gamma \in (1/3, 2/3), \\ c(\gamma - 1/2) & \text{if } \gamma \in (-2/3, -1/3), \end{cases}$$

Therefore, $\widehat{\varphi}(\gamma) = m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2)$, where $m_0(\gamma)$ is the period 1 extension of the function

$$\widehat{\varphi}(2\gamma)\,\mathbf{1}_{[-1/2,1/2]}(\gamma).$$

$\psi(x)$ is defined by

$$\widehat{\psi}(\gamma) = -e^{-\pi i \gamma}\,\overline{m_0(\gamma/2 + 1/2)}\,\widehat{\varphi}(\gamma/2)$$

and

$$\widehat{\psi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \leq 1/3 \text{ or } |\gamma| \geq 4/3, \\ s(\gamma - 1/2) & \text{if } \gamma \in (1/3, 2/3], \\ c(\gamma/2 - 1/2) & \text{if } \gamma \in (2/3, 4/3), \\ s(\gamma/2 + 1/2) & \text{if } \gamma \in (-4/3, -2/3), \\ c(\gamma + 1/2) & \text{if } \gamma \in [-2/3, -1/3). \end{cases}$$

(d) The Piecewise Linear wavelet. Recall that

$$\widehat{\widetilde{\varphi}}(\gamma) = \widehat{\varphi}(\gamma)\,\Phi(\gamma) = \frac{\sqrt{3}\,\widehat{\varphi}(\gamma)}{(1 + 2\,\cos^2(\pi\gamma))^{1/2}},$$

where $\varphi(x) = (1 - |x|)\,\mathbf{1}_{[-1,1]}(x)$ and

$$\Phi(\gamma) = (\sum_n |\widehat{\varphi}(\gamma + n)|^2)^{-1/2}.$$

Also,

$$\widehat{\varphi}(\gamma) = \cos^2(\pi\gamma/2)\,\varphi(\gamma/2).$$

Therefore,

$$\widehat{\widetilde{\varphi}}(\gamma) = \cos^2(\pi\gamma/2)\,(\frac{1 + 2\,\cos^2(\pi\gamma/2)}{1 + 2\,\cos^2(\pi\gamma)})^{1/2}\widehat{\widetilde{\varphi}}(\gamma/2),$$

so that

$$m_0(\gamma) = \cos^2(\pi\gamma)\,(\frac{1 + 2\,\cos^2(\pi\gamma)}{1 + 2\,\cos^2(2\pi\gamma)})^{1/2}.$$

Therefore,

$$m_1(\gamma) = -e^{-2\pi i\gamma} \sin^2(\pi\gamma)(\frac{1+2\sin^2(\pi\gamma)}{1+2\cos^2(2\pi\gamma)})^{1/2}.$$

and

$$\widehat{\psi}(\gamma) = d(\gamma/2)\,\widehat{\varphi}(\gamma/2).$$

where

$$\begin{aligned} d(\gamma) &= -\sqrt{3}\,e^{-\pi i\gamma} \sin^2(\pi\gamma/2) \\ &\quad \times(\frac{1+2\sin^2(\pi\gamma)}{(1+2\cos^2(2\pi\gamma))(1+2\cos^2(\pi\gamma))})^{1/2} \end{aligned}$$

Therefore

$$\psi(x) = \sum_n d(n)\,\varphi_{1,n}(x),$$

where $d(n)$ is the n^{th} Fourier coefficient of $d(\gamma)$.