11-20-03

Purely discrete theory

$h(n)$ - scaling ftn $\leftarrow$ <u>QMF conditions</u>

$g(n) = (-1)^n \overline{h(1-n)}$

H, G, H^*, G^*

$$
\begin{tikzcd}
 & d_1 & & d_2 & & d_3 & \cdots \\
c_0 \arrow[r, "H"'] \arrow[ur, "G"] & c_1 \arrow[rr, "H"'] \arrow[urr, "G"] & & c_2 \arrow[rr, "H"'] \arrow[urr, "G"] & & c_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & d_1 \arrow[dl, "G^*"'] & & d_2 \arrow[dll, "G^*"'] & & d_3 \arrow[dll, "G^*"'] \\
c_0 & c_1 \arrow[l, "H^*"] & & c_2 \arrow[ll, "H^*"] & & c_3 \arrow[ll, "H^*"]
\end{tikzcd}
$$

The DWT as a "sub-band coding scheme"

H – low pass filtering + coding

G – high pass filtering + coding

$\uparrow$ convolution (filtering)

$\uparrow$ decimation or downsampling (coding)

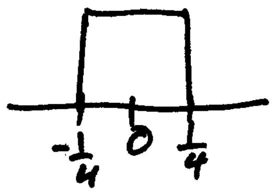
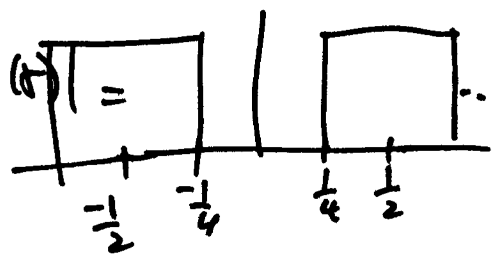
Consider as an example the Bandlimited MRA

$$\begin{tikzcd}
 & c_0(n) \arrow[dl, "H"'] \arrow[dr, "G"] & \\
c_1(n) & & d_1(n)
\end{tikzcd}$$

$$(Hc_0)^\wedge(\gamma) = \frac{1}{\sqrt{2}}\left(\hat{c}_0(\gamma/2)\,\overline{m_0(\gamma/2)} + \hat{c}(\gamma/2+1/2)\,\overline{m_0(\gamma/2+1/2)} \right)$$

$$(H^*Hc_0)^\wedge(\gamma) = \hat{c}(\gamma)\,|m_0(\gamma)|^2 + \hat{c}(\gamma+1/2)\,\overline{m_0(\gamma+1/2)}\,m_0(\gamma).$$

$$(G^*Gc_0)^\wedge(\gamma) = \hat{c}(\gamma)\,|m_1(\gamma)|^2 + \hat{c}(\gamma+1/2)\,\overline{m_1(\gamma+1/2)}\,m_1(\gamma)$$

Rcl: $|m_0(\gamma)| =$  $|m_1(\gamma)| =$ 

$$m_0(\gamma)m_0(\gamma+1/2) = 0 = m_1(\gamma)m_1(\gamma+1/2)$$

$\therefore$ H^*Hc_0 = perfect low pass filter

G^*Gc_0 = " high " "

$\therefore$ c_1 codes the low freq of c_0

d_1 " " high " of c_0.

$$
\begin{tikzcd}[column sep=small]
 & & c_0 \arrow[dl, "H"'] \arrow[dr, "G"] & \\
 & c_1 \arrow[dl, "H"'] \arrow[dr, "G"] & & d_1 \\
 \underline{\underline{c_2}} & & \underline{\underline{d_2}} &
\end{tikzcd}
$$

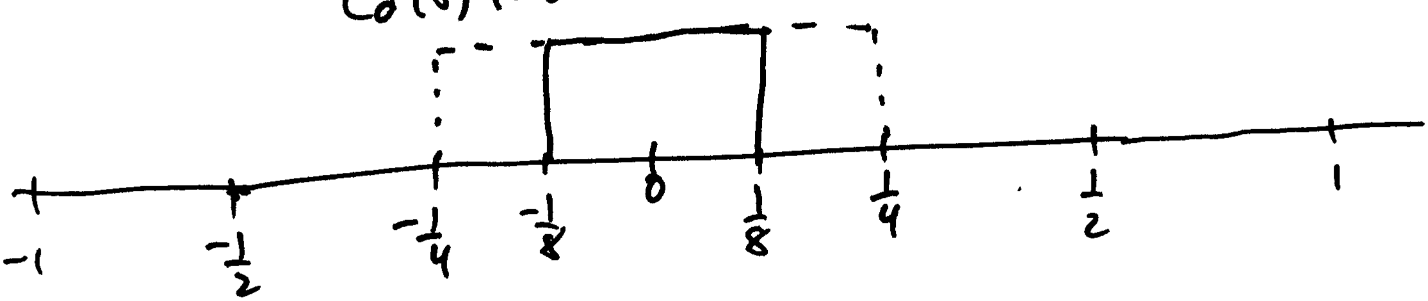
$$(H^*H^*HHc_0)^\wedge(\gamma) = (H^*(H^*H)Hc_0)^\wedge(\gamma)$$

$$= \sqrt{2}\,((H^*H)Hc_0)^\wedge(2\gamma)\,m_0(\gamma)$$

$$= \sqrt{2}\,(Hc_0)^\wedge(2\gamma)\,|m_0(2\gamma)|^2 m_0(\gamma)$$

$$= \hat{c}_0(\gamma)\,\overline{m_0(\gamma)}\,|m_0(2\gamma)|^2 m_0(\gamma)$$

$$= \hat{c}_0(\gamma)\,|m_0(\gamma)|^2|m_0(2\gamma)|^2 = \hat{c}_0(\gamma)\,\chi_{[-1/8,1/8]}$$



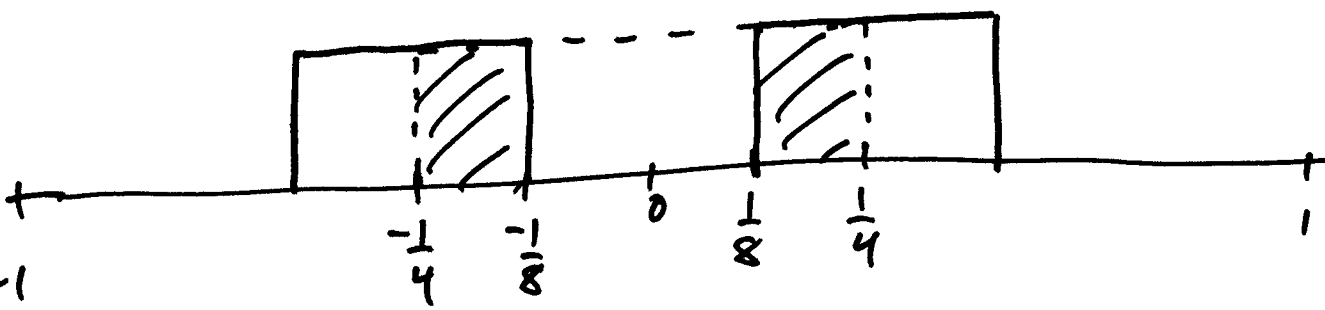
For d_2 we look at:

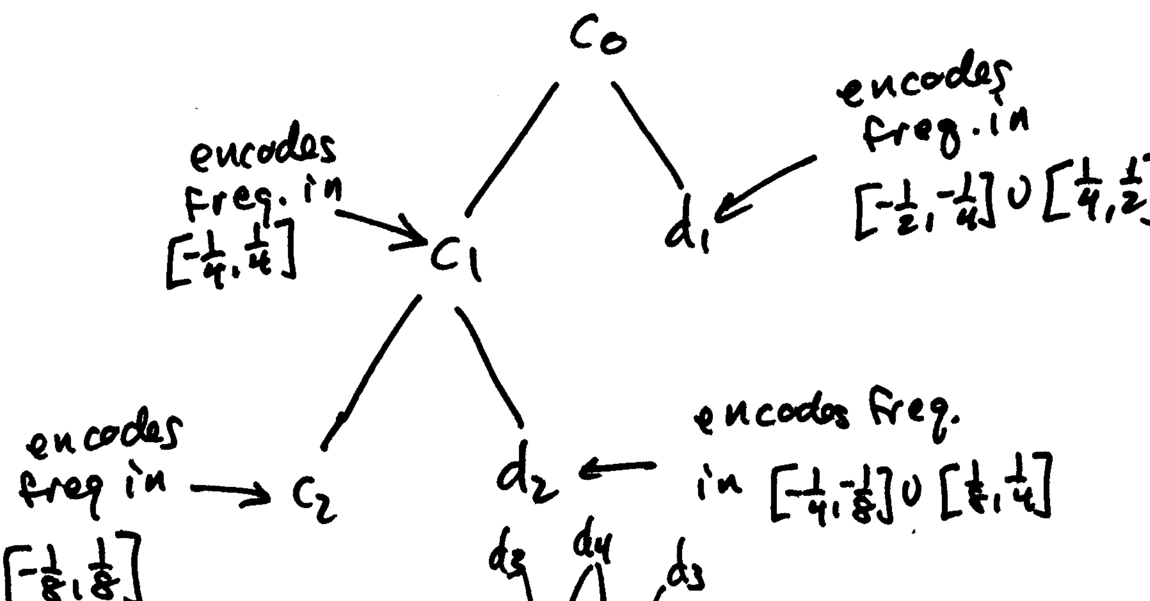
$$(H^*G^*GHc_0)^\wedge(\gamma) = (H^*(G^*G)Hc_0)^\wedge(\gamma)$$

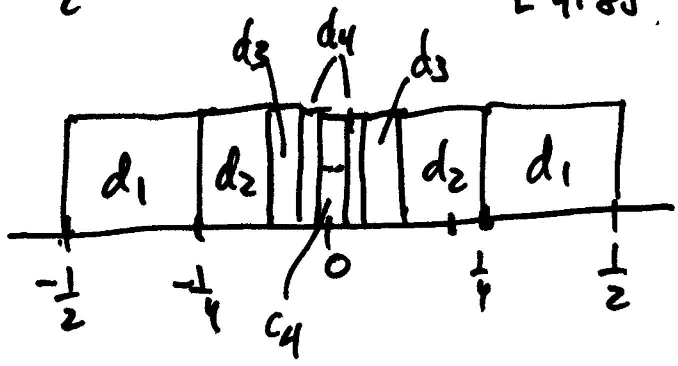
$$= \sqrt{2}\,((G^*G)Hc_0)^\wedge(2\gamma)\, m_0(\gamma)$$

$$= \sqrt{2}\,(Hc_0)^\wedge(2\gamma)\,|m_1(2\gamma)|^2 m_0(\gamma)$$

$$= \hat{c}_0(\gamma)\,|m_0(\gamma)|^2\,|m_1(2\gamma)|^2$$







Variation #1 : Biorthogonal wavelets

$$\begin{tikzcd}
& & c_0 \arrow[dl, "H"'] \arrow[dr, "G"] & \\
& c_1 \arrow[ur, bend left, "\tilde{H}^*"] \arrow[dl, "H"'] \arrow[dr, "G"] & & d_1 \arrow[ul, bend right, "\tilde{G}^*"'] \\
c_2 \arrow[ur, bend left, "\tilde{H}^*"] & & d_2 \arrow[ul, bend right, "\tilde{G}^*"'] &
\end{tikzcd}$$

Start with $h(n)$, $\tilde{h}(n)$

Define $g(n) = (-1)^n \overline{\tilde{h}(1-n)}$, $\tilde{g}(n) = (-1)^n \overline{h(1-n)}$

We want perfect reconstruction, i.e.,

$$\tilde{H}^* H + \tilde{G}^* G = I$$

Also would like "biorthogonality", i.e.,

$$H \tilde{H}^* = G \tilde{G}^* = I$$

These are equivalent to

$$\sum_n h(n) \overline{\tilde{h}(n-2k)} = \delta(k)$$

$$\sum_k \overline{h(m-2k)}\, \tilde{h}(n-2k) + \sum_k g(m-2k)\, \overline{\tilde{g}(n-2k)} = \delta(m-n)$$

This reduces to (like before)

$$m_0(\gamma) \overline{\tilde{m}_0(\gamma)} + m_0(\gamma + 1/2) \overline{\tilde{m}_0(\gamma+1/2)} = 1$$

where $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum h(n) e^{-2\pi i n \gamma}$

$$\tilde{m}_0(\gamma) = \frac{1}{\sqrt{2}} \sum \tilde{h}(n) e^{-2\pi i n \gamma}$$

we also ask that $m_0(0) = \tilde{m}_0(0) = 1$.

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we can design biorth filters similarly to before:

$$(1-y)^{2N} P_{N-1}(y) + y^{2N} P_{N-1}(1-y) = 1$$

Before we wanted $|m_0(\gamma)|^2 = (1-\sin^2 \pi\gamma)^{2N} P_{N-1}(\sin^2 \pi\gamma)$

$$= P_{2N-1}(e^{2\pi i \gamma})$$

$\therefore$ $m_0(\gamma)$ was arrived at by factoring $P_{2N-1}(z)$.

Now all we need to do is find $m_0(\gamma)$ + $\tilde{m}_0(\gamma)$ s.t.

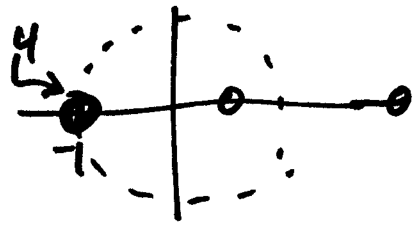
$$m_0(\gamma)\,\tilde{m}_0(\gamma) \cong P_{2N-1}(z)$$

Recall: $P_{2N-1}(z) = (z^{-1}+1)^{2N} Q(z)$.

This allows for designing symmetric wavelet filters.

e.g Take $N=2$ $\quad 2N-1=3$

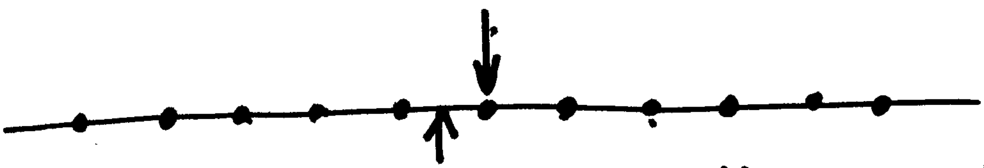
$$P_3(z) = \frac{-1}{32}(z^{-1}+1)^4(z^2-4z+1)$$



Design for symmetry.

By symmetry I mean:

For some M, $\quad h(M-n) = h(n)$



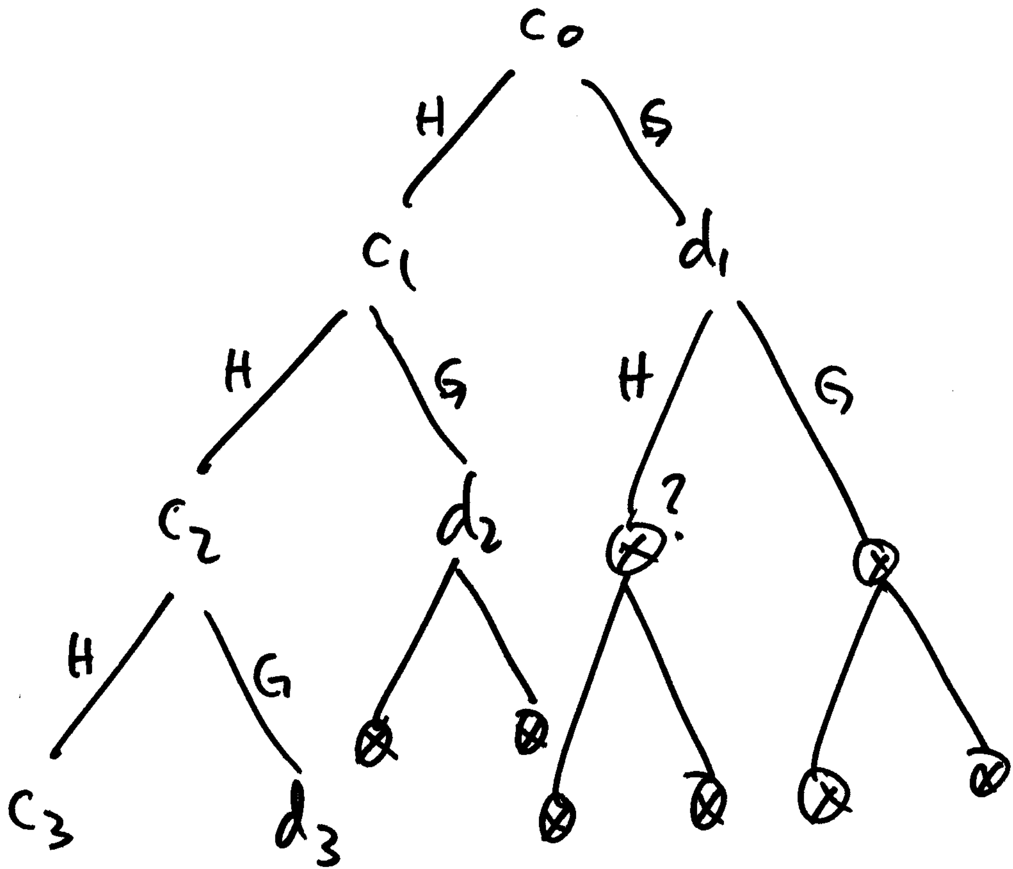
If I reflect h about $\frac{M}{2}$ then it does not change.

M even – whole point symmetry

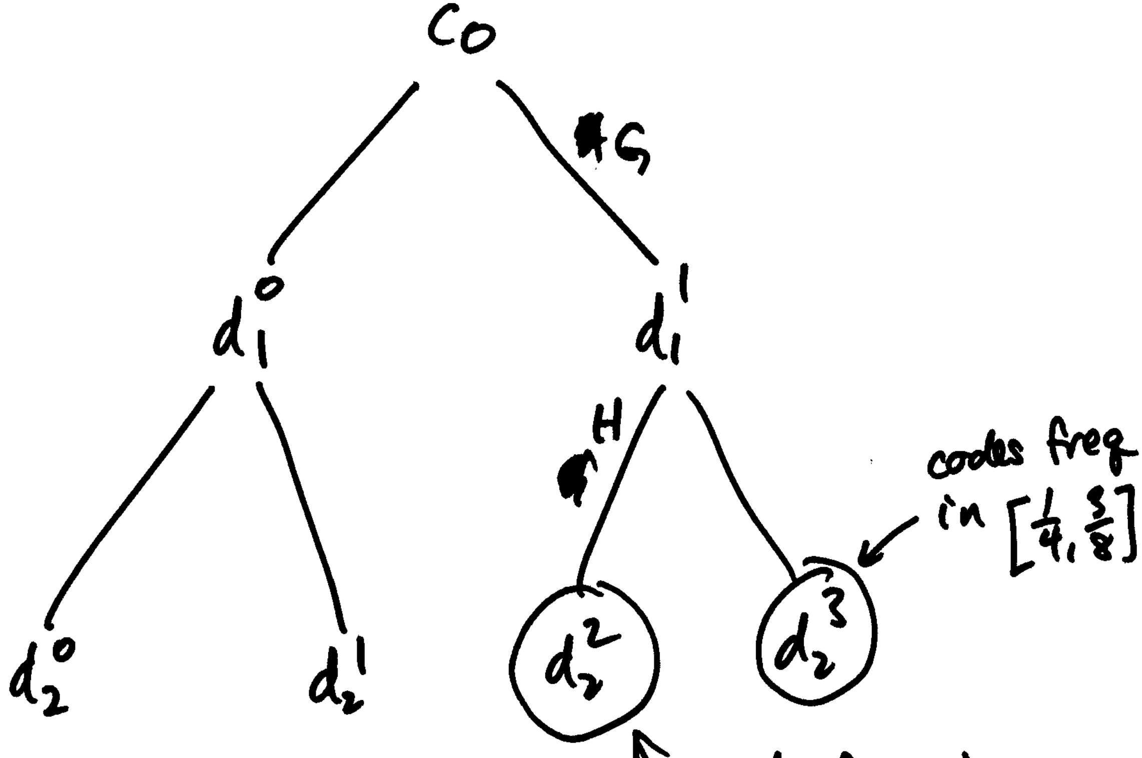
M odd – half point symmetry.

Having symmetry in $h(n)$ turns out to be equivalent to factoring P_{2N-1} into terms of the form $c(z+1)^m \prod_{k=1}^{K}(z-z_k)(z-z_k^{-1})$

wavelet packets



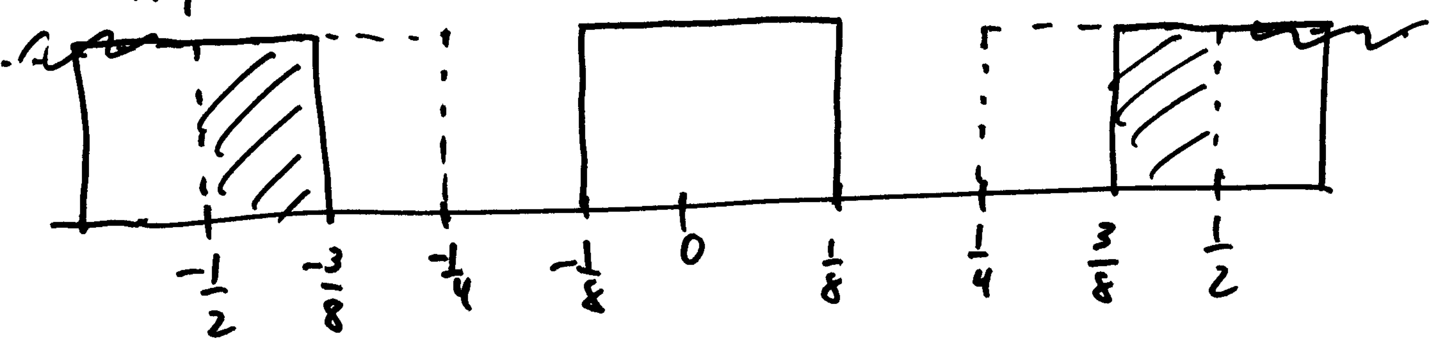
Rename the tree

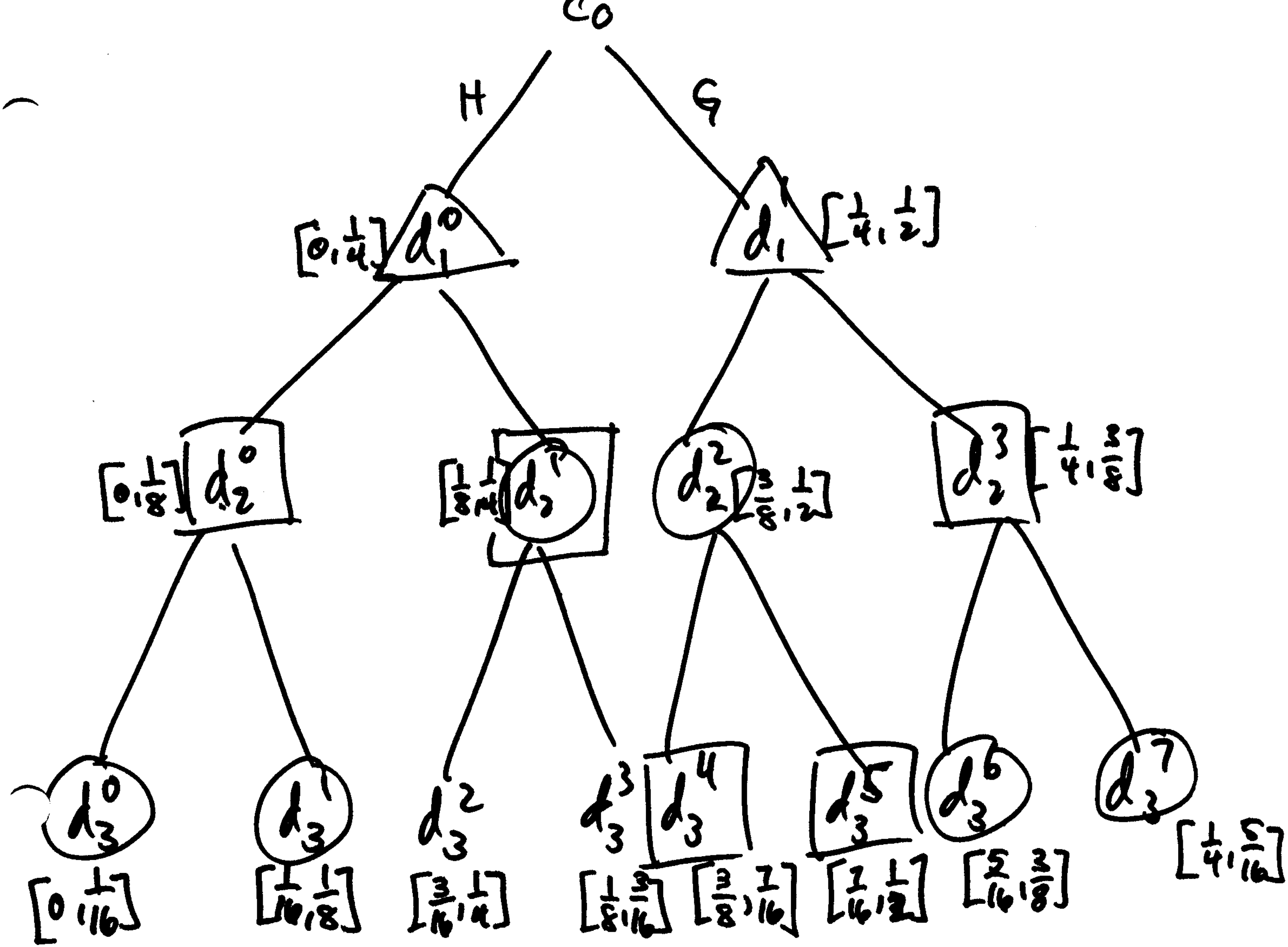


What is d_2^2?

$$(G^* H^* H G c_0)^\wedge(\gamma) = \hat{c}_0(\gamma)\, |m_1(\gamma)|^2\, |m_0(2\gamma)|^2$$

Again assuming Band limited MRA:





c_0
H
G
$[0, \frac{1}{4}]$ d_1^0
d_1 $[\frac{1}{4}, \frac{1}{2}]$
$[0, \frac{1}{8}]$ d_2^0
$[\frac{1}{8}, \frac{1}{4}]$ d_2^1
d_2^2 $[\frac{3}{8}, \frac{1}{2}]$
d_2^3 $[\frac{1}{4}, \frac{3}{8}]$
d_3^0 $[0, \frac{1}{16}]$
d_3^1 $[\frac{1}{16}, \frac{1}{8}]$
d_3^2 $[\frac{3}{16}, \frac{1}{4}]$
d_3^3 $[\frac{1}{8}, \frac{3}{16}]$
d_3^4 $[\frac{3}{8}, \frac{7}{16}]$
d_3^5 $[\frac{7}{16}, \frac{1}{2}]$
d_3^6 $[\frac{5}{16}, \frac{3}{8}]$
d_3^7 $[\frac{1}{4}, \frac{5}{16}]$