11-13-03

MRA

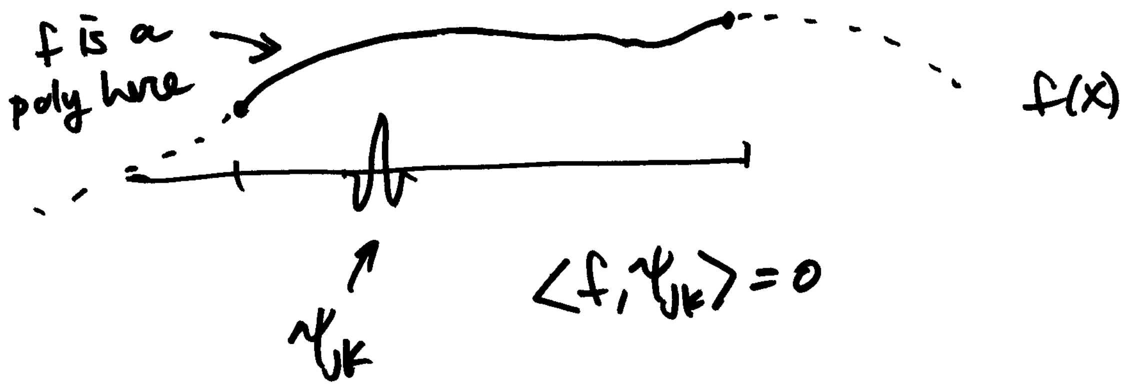
vanishing moments of ψ.
- get one automatically ($\int \psi = 0$)
- what about higher moments?

Desirable properties related to vanishing moments:

a. smoothness (ψ smooth $\Longrightarrow$ ψ has v. moments)

b. approximation (if ψ has N v. moments then $|\langle f, \psi_{jk}\rangle| = O(2^{-j(N+1/2)})$ if f is C^N).

c. reproduction of polynomials (if ψ has N v.m. ~~then~~ + cpt support then any poly of degree $N-1$ can be written as a sum of scaling ftns)



How to design a wavelet with vanishing moments?

Start with the scaling filter ....

Pf (a) $\Leftrightarrow$ (b)

$$\int x^k \psi(x)\,dx = 0 \iff \hat{\psi}^{(k)}(0) = 0$$

$$\hat{\psi}(\gamma) = m_1(\gamma/2)\hat{\varphi}(\gamma/2) = e^{-i\pi(\gamma/2+1/2)}\,\overline{m_0(\gamma/2+1/2)}\,\hat{\varphi}(\gamma/2)$$

$$\hat{\psi}(0) = e^{-i\pi/2}\,\overline{m_0(1/2)}\,\hat{\varphi}(0)$$

$$\hat{\psi}(0) = 0 \iff m_0(1/2) = 0$$

$$\hat{\psi}'(\gamma) = \underbrace{\overline{m_0(\gamma/2+1/2)}}_{=0 \text{ at } \gamma=0}\frac{d}{d\gamma}(\text{—}) + \frac{1}{2}\overline{m_0'(\gamma/2+1/2)}\underbrace{(\text{—})}_{\neq 0 \text{ at } \gamma=0}$$

$$\hat{\psi}'(0) = 0 \iff m_0'(1/2) = 0 \qquad \text{etc} \cdots$$

(b) $\Leftrightarrow$ (c) $\quad m_0(\gamma) = \frac{1}{\sqrt{2}}\sum_n h(n)e^{-2\pi i n\gamma}$

Let $A(z) = \frac{1}{\sqrt{2}}\sum_{n=0}^{2N-1} h(n) z^{-n}$ so $A(e^{2\pi i\gamma}) = m_0(\gamma)$

Assume A is a poly in z^{-1}! $\gamma = \frac{1}{2} \Rightarrow z^{-1} = -1$

$m_0^{(k)}(1/2) = 0 \iff A^{(k)}(-1) = 0$ for $0 \le k \le N-1$

$\therefore A(z) = (1+z^{-1})^N R(z^{-1})$ ← polynomial

$\therefore m_0(\gamma) = (1+e^{-2\pi i\gamma})^N R(e^{-2\pi i\gamma})$

$= \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma).$

$\left[\; P(x) - \text{poly} \quad P(x_0) = 0 \right.$

$\quad P(x) = (x - x_0)\,Q(x)$

$\quad$ etc $\cdots \;\Big]$

(b) $\Leftrightarrow$ (d) $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_n h(n) e^{-2\pi i n \gamma}$

$$m_0'(\gamma) = \frac{-2\pi i}{\sqrt{2}} \sum_n n\, h(n) e^{-2\pi i n \gamma}$$

$$m_0'(\tfrac{1}{2}) = \frac{-2\pi i}{\sqrt{2}} \sum_n n\, h(n) \underbrace{e^{-\pi i n}}_{(-1)^n}$$

$$\therefore m_0^{(k)}(\tfrac{1}{2}) = \frac{(-2\pi i)^k}{\sqrt{2}} \sum_n n^k h(n) (-1)^n$$

We want to find $m_0(\gamma)$ s.t.

1. $m_0(\gamma)$ is a trig poly ($h(n)$ finite)
2. $m_0(\gamma) = \left(\frac{1+e^{-2\pi i \gamma}}{2}\right)^n \mathcal{L}(\gamma)$
3. $|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2 = 1$; $m_0(0) = 1$

$$m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma)$$

$$|m_0(\gamma)|^2 = \underbrace{\left|\frac{1+e^{-2\pi i\gamma}}{2}\right|}^{2N} |\mathcal{L}(\gamma)|^2$$

$$e^{-\pi i\gamma}\left(\frac{e^{\pi i\gamma}+e^{-\pi i\gamma}}{2}\right) = e^{-\pi i\gamma}\cos(\pi\gamma)$$

$$= \cos^{2N}(\pi\gamma)\ \underbrace{L(\gamma)}_{\text{trig poly.}}$$

(c) $L(\gamma) = \sum_n c(n) e^{-2\pi i n\gamma}$ $c(n)$ real.

$$\overline{L(\gamma)} = \sum_n c(n) e^{2\pi i n\gamma} = L(\gamma) = \sum_n c(-n) e^{2\pi i n\gamma}$$

$\therefore c(n) = c(-n)$

$$\therefore L(\gamma) = c(0) + \sum_{n>0} c(n) e^{-2\pi i n\gamma} + \sum_{n<0} c(n) e^{-2\pi i n\gamma}$$

$$= c(0) + \sum_{n>0} 2c(n)\left(\frac{e^{-2\pi i n\gamma} + e^{2\pi i n\gamma}}{2}\right)$$

$$= c(0) + \sum_{n>0} 2c(n)\cos(2\pi n\gamma)$$

$$\cos n\alpha = 2\cos(n-1)\alpha \cdot \cos\alpha - \cos(n-2)\alpha$$

$$\therefore \cos(2\pi n\gamma) = \text{poly in } \cos 2\pi\gamma$$

$$= \text{poly in } (1-\sin^2\pi\gamma) \leftarrow \text{double angle formula}$$

$$= \text{poly in } \sin^2\pi\gamma$$

$$\boxed{L(\gamma) = P(\sin^2\pi\gamma)}$$

What does P satisfy? QMF cond.

$$|m_0(\gamma)|^2 = \underbrace{\cos^{2N}(\pi\gamma)}_{(1-\sin^2\pi\gamma)^N} P(\sin^2\pi\gamma)$$

$$|m_0(\gamma+1/2)|^2 = \cos^{2N}(\pi\gamma+\pi/2)\, P(\sin^2(\pi\gamma+\pi/2))$$

$$= \sin^{2N}(\pi\gamma)\, P(1-\sin^2\pi\gamma)$$

$$\therefore 1 = (1-\sin^2\pi\gamma)^N P(\sin^2\pi\gamma) + (\sin^2\pi\gamma)^N P(1-\sin^2\pi\gamma)$$

or with $y = \sin^2\pi\gamma$

$$1 = (1-y)^N P(y) + y^N P(1-y) \quad y \in [-1,1]$$

$$\left[\text{use } 1 = (1-y+y)^{2N}\right]$$

Now have

$$1 = (1-y)^N P_{N-1}(y) + y^N P_{N-1}(1-y)$$

Idea: $|m_0(\gamma)|^2 = \underbrace{(1-\sin^2\pi\gamma)^N}_{\left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^{2N}} \underbrace{P_{N-1}(\sin^2\pi\gamma)}_{L(\gamma)}$

$$L(\gamma) = |\mathcal{L}(\gamma)|^2$$

Need to back out $\mathcal{L}(\gamma)$ from $P_{N-1}(\sin^2\pi\gamma)$

Then $m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma)$

$\underline{N=1}$

$$P_0(y) = 1$$

$$P_0(\sin^2\pi\gamma) = 1$$

$$|\mathcal{L}(\gamma)|^2 = 1 \quad \therefore \mathcal{L}(\gamma) = 1$$

$$\therefore m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right) = \frac{1}{2} + \frac{1}{2}e^{-2\pi i\gamma}$$

$$= \frac{1}{\sqrt{2}} \sum h(n) e^{-2\pi i n\gamma}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e^{-2\pi i\gamma}\right)$$

$$\therefore h(n) = \begin{cases} \frac{1}{\sqrt{2}} & n = 0,1 \\ 0 & n \neq 0,1 \end{cases}$$

Haar!

$\underline{N=2}$ $\quad P_1(y) = 1+2y$

$$P_1(\sin^2\pi\gamma) = 1 + 2\sin^2\pi\gamma$$

$$P_1(\sin^2\pi\gamma) = |\mathcal{L}(\gamma)|^2$$

$$\mathcal{L}(\gamma) = a + b\,e^{-2\pi i\gamma}$$

$$|\mathcal{L}(\gamma)|^2 = (a+b\,e^{-2\pi i\gamma})(a+b\,e^{2\pi i\gamma})$$

$$= (a^2+b^2) + ab\,(e^{-2\pi i\gamma} + e^{2\pi i\gamma})$$

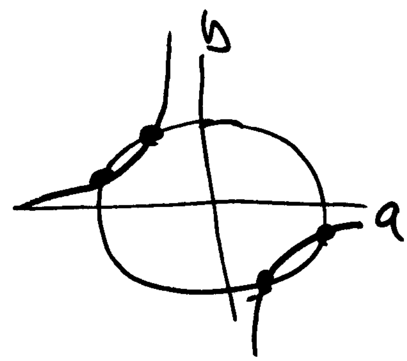
$$= (a^2+b^2) + 2ab\cos 2\pi\gamma$$

Note: $\sin^2\pi\gamma = \frac{1}{2} - \frac{1}{2}\cos 2\pi\gamma$

$\therefore P_1(\sin^2\pi\gamma) = 2 - \cos 2\pi\gamma$

$\therefore a^2+b^2 = 2$

$\quad 2ab = -1$



add the condition $\underline{a+b=1}$ ($\Leftrightarrow \underline{m_0(0)=1}$)

ultimately $$m_0(\gamma) = \underbrace{\left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^2}_{\cancel{4\ \text{terms}}} \underbrace{\mathcal{L}(\gamma)}_{\cancel{2\ \text{terms}}}.$$

$=$ degree 4 trig poly

$P_{N-1}(\sin^2 \pi\gamma) = |\mathcal{L}(\gamma)|^2$

Need to find $\mathcal{L}(\gamma)$ from this.

$P_{2N-1}(y) = (1-y)^N P_{N-1}(y)$

$P_{2N-1}(\sin^2 \pi\gamma) =$ poly in $e^{2\pi i\gamma}$

$$\sin^2 \pi\gamma = \frac{1}{2} - \frac{1}{2}\cos 2\pi\gamma = \frac{1}{2} - \frac{1}{2}\left(\frac{e^{2\pi i\gamma} + e^{-2\pi i\gamma}}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{4}\left(e^{2\pi i\gamma} + e^{-2\pi i\gamma}\right)$$

Let $z = e^{2\pi i\gamma}$

$$= \frac{1}{2} - \frac{1}{4}\left(z + z^{-1}\right)$$

$P_{2N-1}(\sin^2 \pi\gamma) = \mathcal{P}_{2N-1}(e^{2\pi i\gamma})$

some poly $\mathcal{P}_{2N-1}(z)$

Theorem. For each $N \in \mathbf{N}$, $\mathbf{P}_{2N-1}(z)$ satisfies:

(a) $\mathbf{P}_{2N-1}(z) = \sum_{m=-2N+1}^{2N-1} a_m z^m$ for some real-valued coefficients a_m.

(b) $\mathbf{P}_{2N-1}(z) + \mathbf{P}_{2N-1}(-z) = 1$ for all $z \in \mathbf{C}$, $z \neq 0$. Follows from $1 = (1-y)^N P_{N-1}(y) + y^N P_N(1-y)$

(c) $\mathbf{P}_{2N-1}(z) \geq 0$ for $|z| = 1$.

→ (d) $\mathbf{P}_{2N-1}(z) = \mathbf{P}_{2N-1}(z^{-1})$ for all $z \in \mathbf{C}$, $z \neq 0$. Follows from fact that $P_{2N-1}(z) = \text{poly in } \left(\frac{1}{2}+\frac{1}{4}(z+z^{-1})\right)$

(e) $a_m = a_{-m}$ for $-2N+1 \leq m \leq 2N-1$.

(f) $a_m = 0$ if m is even and $m \neq 0$, and $a_0 = 1/2$.

Examples.

4 coeff scaling ~~for~~ filter

(a) With $N = 2$,

$$\widetilde{\mathrm{P}}_6(z) = \frac{1}{32}(-1 + 9z^2 + 16z^3 + 9z^4 - z^6).$$

We factor

$$\begin{aligned}\widetilde{\mathrm{P}}_6(z) &= \frac{1}{32}(z+1)^4(-z^2+4z-1) \\ &-\frac{1}{32}(z+1)^4(z-(2-\sqrt{3}))(z-(2+\sqrt{3})).\end{aligned}$$

Therefore,

$$\begin{aligned}B_3(z) &= \overset{|\alpha|^{1/2}}{\frac{1}{4\sqrt{2}}}(z+1)^2\overbrace{(2-\sqrt{3})}^{|z_0|^{-1/2}}{}^{-1/2}(z-\overbrace{(2-\sqrt{3})}^{z_0}) \\ &= \frac{1+\sqrt{3}}{8}(z+1)^2(z-(2-\sqrt{3})) \\ &= \frac{1+\sqrt{3}}{8}z^3 + \frac{3+\sqrt{3}}{8}z^2 + \frac{3-\sqrt{3}}{8}z + \frac{1-\sqrt{3}}{8}.\end{aligned}$$

$z = e^{2\pi i\gamma}$

$$B_3(e^{2\pi i\gamma}) = m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^2 \mathcal{L}(\gamma)$$

$$
\begin{tikzcd}
 & c_0(n) \arrow[dl, "H"'] \arrow[dr, "G"] & \\
c_1(n) \arrow[ur, bend left, "H^*"] & & d_1(n) \arrow[ul, bend right, "G^*"']
\end{tikzcd}
$$

$$H = \underbrace{\text{convolution}}_{\substack{m_0(\gamma) \\ \text{low pass}}} \rightarrow \underbrace{\text{downsampling}}_{\text{error}}$$

$$G = \underset{\substack{m_1(\gamma) \\ \text{high pass.}}}{\text{convolution}} \rightarrow \underbrace{\text{downsampling}}_{\text{erro}}$$

$H^* =$ upsample $\rightarrow$ convolution

$G^* =$ upsample $\rightarrow$ convolution

$H^* H c_0 =$ "true" low pass version of c_0
+ some error

$G^* G c_0 =$ "true" high pass version of c_0
+ error