11-6-03

MRA

wavelet recipe

MRA $\to$ φ scaling ftn $\to$ scaling filter $h(n)$

$\to$ wavelet filter $g(n)$ $\to$ wavelet ψ.

DWT

$$\begin{tikzcd}
 & d_1(k) & d_2(k) & & \cdots & d_J \\
c_0(k) \arrow[ur, "G"] \arrow[r, "H"'] & c_1(k) \arrow[ur, "G"] \arrow[r, "H"'] & c_2(k) \arrow[urr, "G"] \arrow[rr, "H"'] & & \cdots \arrow[ur, ""] \arrow[r, "H"'] & c_J
\end{tikzcd}$$

$$\begin{tikzcd}
 & d_1 \arrow[dl, "G^*"'] & & & & d_{J-1} \arrow[dl, "G^*"'] & & d_J \arrow[dl, "G^*"'] \\
c_0 & c_1 \arrow[l, "H^*"] & \arrow[l] \arrow[ul] & \cdots & c_{J-2} & & c_{J-1} \arrow[ll, "H^*"] & c_J \arrow[l, "H^*"]
\end{tikzcd}$$

Point: Define the DWT entirely in terms of scaling filter $h(n)$, <u>never</u> mention the scaling function $\varphi(x)$.

Rewrite all relevant properties of MRA in terms of $h(n)$.

Boils down to the QMF conditions:

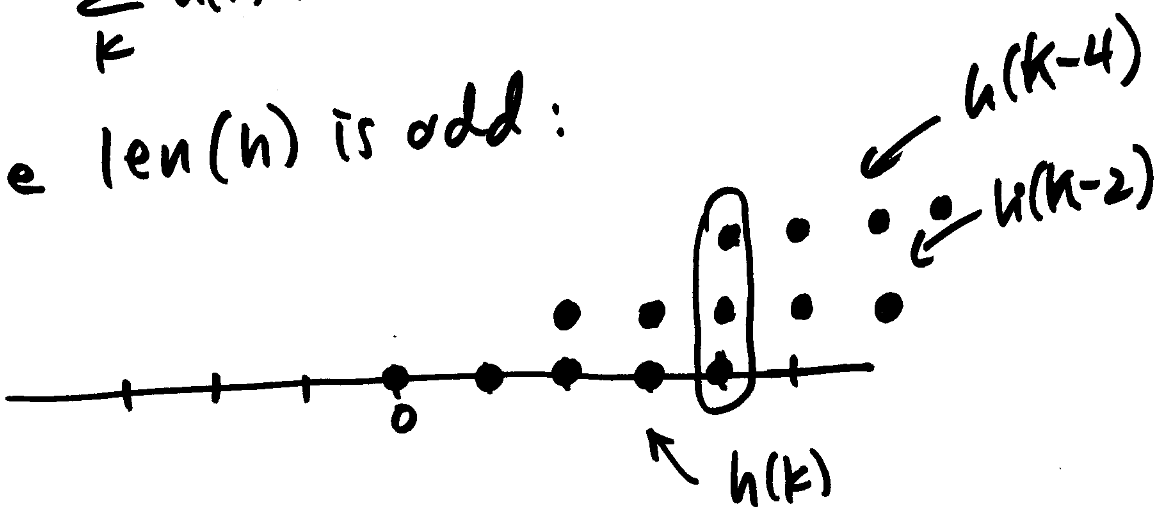
1. $m_0(0) = 1$
2. $|m_0(\gamma)|^2 + |m_0(\gamma + \frac{1}{2})|^2 = 1$

h(n) satisfies QMF $\Rightarrow$ Length h(n) is even.

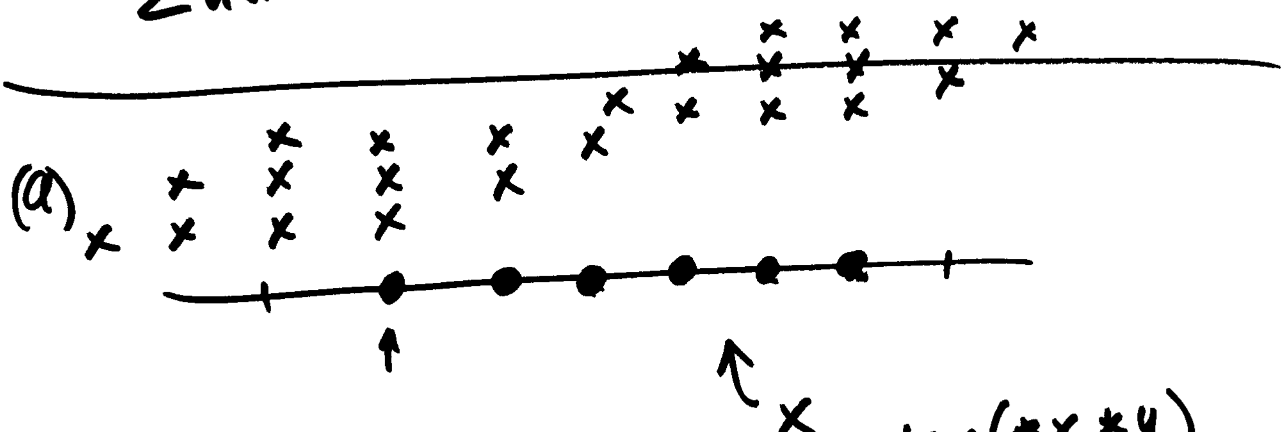
<u>Idea</u>: One of <u>QMF conds</u> is

$$\sum_k h(k)\overline{h(k-2n)} = \delta(n)$$

Spse len(h) is odd:

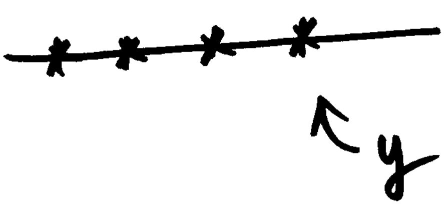


$$\sum h(k)\overline{h(k-2n)} = h(4)\,h(0) = 0$$



$\text{len}(x * y)$

$= \text{len}(x) + (\text{len}(y) - 1)$



(b) c_1, d_1 have length $2^{N-1} + \left(\frac{L-2}{2}\right)$

c_2, d_2 have length

$$\frac{2^{N-1} + \left(\frac{L-2}{2}\right) + (L-2)}{2} = 2^{N-2} + \frac{L-2}{2} + \frac{L-2}{4}$$

$$= 2^{N-2} + \frac{3}{4}(L-2)$$

MATLAB illustration.

```
>> x=[0 1 2 3 4 5 6 7 8 7 6 5 4 3 2 1];
```

length(x) = 16

```
>> dwtmode('zpd')

****************************************
**  DWT Extension Mode: Zero Padding  **
****************************************

>> [h g h1 g1]=wfilters('db2');
>> h
h =
   -0.1294    0.2241    0.8365    0.4830
```

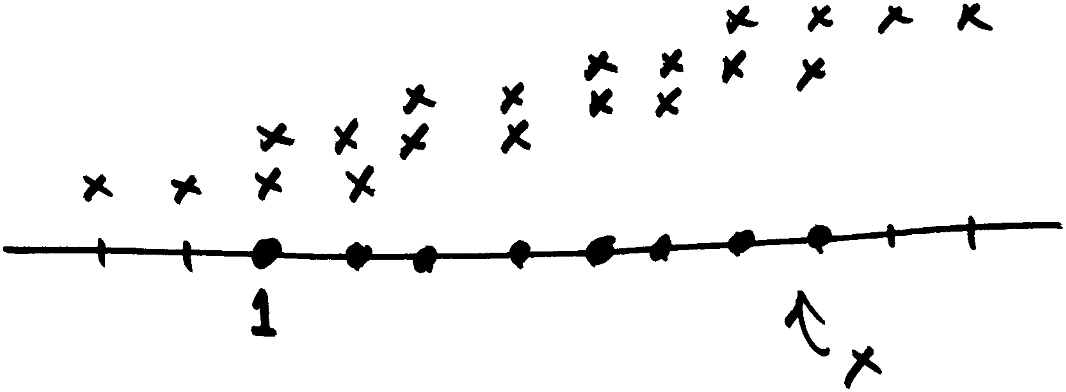
length 4

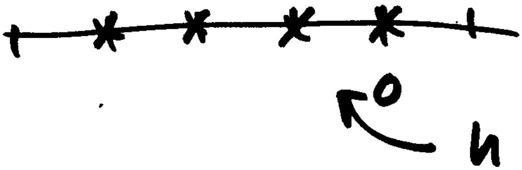
```
>> [c1 d1]=dwt(x,'db2')
c1 =
   -0.1294    0.8966    3.7250    6.5534    9.6407
   10.4171    7.5887    4.7603    1.8024
d1 =
   -0.4830   -0.0000   -0.0000   -0.0000    0.9659
   0.0000    0.0000    0.0000   -0.4830
```

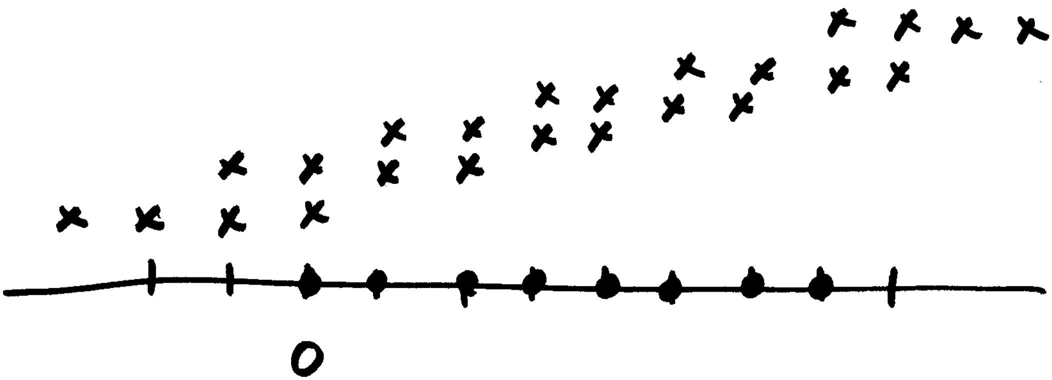
length $\frac{2^N + (L-2)}{2}$

```
>> length(x)
ans =
    16
>> length([c1 d1])
ans =
    18
>> [C L]=wavedec(x,4,'db2');
>> length(C)
ans =
    25
```

length $\underbrace{2^N + J(L-2)}_{16 + 4\cdot 2 = 24} + 1$



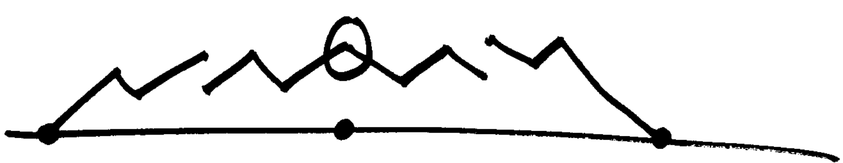




For zero padding – length depends on where the $c_0(k)$ "starts" why have different periodization/extension modes?

spse x looks like:

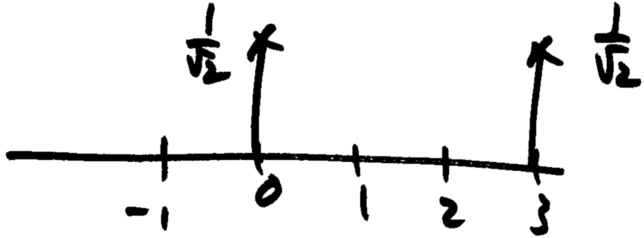
introduce jump if in per mode



symmetric extension avoids this.

example

$h(n)$:



Why is it a QMF?

$$\sum h(k)h(k-2n) = \delta(n) \text{ holds.}$$

This does *not* correspond to any scaling function φ in $L^2(\mathbb{R})$.

we know:

1. $\prod_{j=1}^{\infty} m_0(\gamma/2^j)$ converges in L^∞ on $[-R,R]$ ~~whenever~~ for every $R>0$ and assuming $h(n)$ is finite.

2. So we <u>define</u> $\hat{\varphi}(\gamma) = \prod_{j=1}^{\infty} m_0(\gamma/2^j)$

Q: is $\hat{\varphi}$ in $L^2(\mathbb{R})$?

is $\{T_n \varphi\}$ an o.n. system?

If we look at a partial product

$\prod_{j=1}^{n} m_0(\gamma/2^j)$, it has period 2^n

So a periodic function can <u>never</u> be close to an L^2 function.

Note:
$$\hat{\eta}_\ell(\gamma) = m_0(\gamma/2)\hat{\eta}_{\ell-1}(\gamma/2)$$
$$= m_0(\gamma/2)m_0(\gamma/4)\hat{\eta}_{\ell-2}(\gamma/4)$$
$$= \cdots$$
$$= \prod_{j=1}^{\ell} m_0(\gamma/2^j)\,\hat{\eta}_0(\gamma/2^\ell)$$
$$= \prod_{j=1}^{\ell} m_0(\gamma/2^j)\,\frac{\sin \pi\gamma/2^\ell}{\pi\gamma/2^\ell}$$
$$= \underbrace{F(\gamma)}_{\uparrow\ \text{period } 2^\ell}\,\frac{\sin \pi\gamma/2^\ell}{\pi\gamma/2^\ell}$$

Dilate by 2^ℓ and look at $\underbrace{F(2^\ell\gamma)}_{\text{per } 1}\,\dfrac{\sin\pi\gamma}{\pi\gamma}$

$$\therefore\ \eta_\ell(\gamma 2^{-\ell})\,2^{-\ell/2} = \sum_k c(k)\,1_{[-1/2,1/2]}(\gamma-k)$$

$c(k)$ = Fourier coeff of $F(2^\ell\gamma)$
= Fourier coeff of $m_0(\gamma)m_0(2\gamma)m_0(4\gamma)\cdots m_0(2^\ell\gamma)$

Recall that $(H^*c)^\wedge(\gamma) = \sqrt{2}\, m_0(\gamma)\,\hat{c}(2\gamma)$

$$((H^*)^2 c)^\wedge(\gamma) = (\sqrt{2})^2\, m_0(\gamma)\, m_0(2\gamma)\,\hat{c}(4\gamma)$$

$$\vdots$$

$$[(H^*)^\ell c]^\wedge(\gamma) = 2^{\ell/2}\, m_0(\gamma)\, m_0(2\gamma) \cdots m_0(2^{\ell-1}\gamma)\,\hat{c}(2^\ell \gamma)$$

Spse I start with $c = \delta$

$$\hat{c}(\gamma) = 1$$

$$\therefore [(H^*)^\ell \delta]^\wedge(\gamma) = 2^{\ell/2} \prod_{j=0}^{\ell-1} m_0(\gamma/2^j)$$

$$\therefore 2^{-\ell/2} (H^*)^\ell \delta\,(k) = \underline{c}(k) \text{ from before}$$

$$\left.\begin{tikzcd} & H^*x & \\ x \arrow[ur, "H^*"] & & 0 \arrow[ul, "G^*"'] \end{tikzcd}\right\}\ \text{idwt}$$

$[0001000]$ $\quad$ $[0000000]$

Given a function $f(x)$, its n^{th} moment is

$$\int_{-\infty}^{\infty} x^n f(x)\, dx$$

We have seen that a wavelet ψ satisfies

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0$$

, i.e. its zeroth moment vanishes. We want to design wavelets $\psi(x)$ with more vanishing moments. Why?

1. Smoothness
2. Approximation
3. Reproduction of Polynomials.

Vanishing Moments.

Relation to Smoothness.

Theorem. Suppose that $\{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is an orthogonal system on $\mathbf{R}$ and that $\psi(x)$ and $\widehat{\psi}(\gamma)$ are both L^1 on $\mathbf{R}$. Then $\int_{\mathbf{R}} \psi(x)\,dx = 0$.

smooth says ψ is cont.

Theorem. Let $\psi(x)$ be such that for some $N \in \mathbf{N}$, both $x^N\psi(x)$ and $\boxed{\gamma^{N+1}\widehat{\psi}(\gamma) \text{ are in } L^1(\mathbf{R})}$. If $\{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is an orthogonal system on $\mathbf{R}$, then $\int_{\mathbf{R}} x^m\,\psi(x)\,dx = 0$ for $0 \le m \le N$.

says that ψ is C^N on $\mathbb{R}$

Idea of proof:

$\{\psi_{jk}\}$ is an orthogonal system.

Want to show $\hat{\psi}(0)=0$

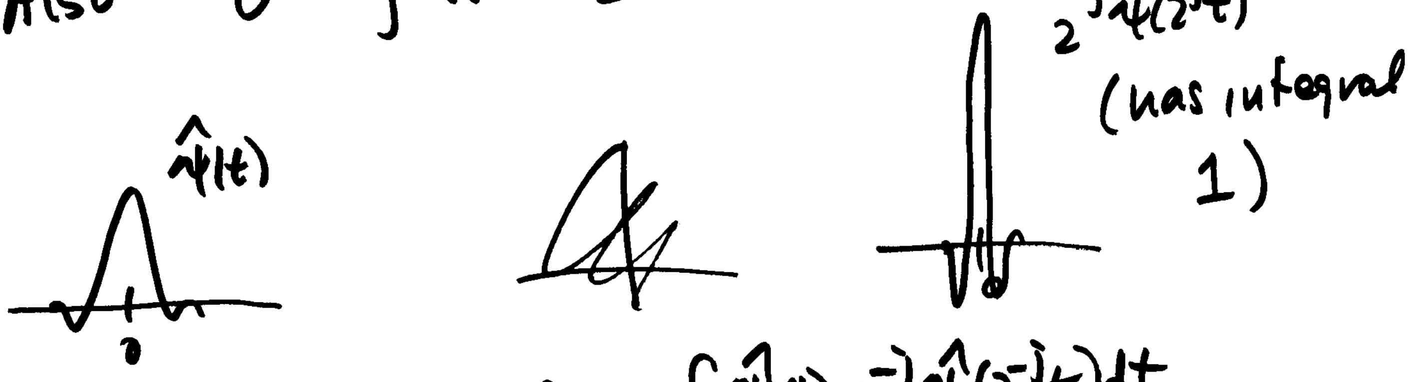
Look at $\langle \psi, \psi_{j,0} \rangle$ and spse $\int \hat{\psi} = 1$, i.e. $\hat{\psi}(0) = 1$.

By orthogonality, $0 = \langle \hat{\psi}_{00}, \hat{\psi}_{j0} \rangle$

$$= \int \hat{\psi}(t)\, 2^{-j/2} \hat{\psi}(2^{-j}t)\, dt$$

$$= \int \hat{\psi}(t)\, 2^{-j} \hat{\psi}(2^{-j}t)\, dt.$$

Also $\quad 0 = \int \hat{\psi}(t)\, 2^{-j} \hat{\psi}(2^{-j}t)\, dt$



Let $j \to -\infty$, $\quad 0 = \int \hat{\psi}(t) \underbrace{2^{-j}\hat{\psi}(2^{-j}t)}_{\delta(t)} dt$

$$\longrightarrow \hat{\psi}(0)$$

Idea of proof: If $f \in C^N(\mathbb{R})$ and ψ has compact support: $\int f(x)\psi_{JK}(x)\,dx$

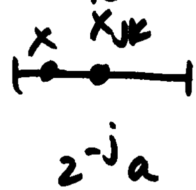
Look at Taylor expansion of $f(x)$ on the interval ~~sup~~ of supp of ψ_{JK}.

$$f(x) = \left.\begin{aligned} & f(\tilde{x}_{JK}) + f'(\tilde{x}_{JK})(x-\tilde{x}_{JK}) \\ & + \cdots + \frac{1}{(N-1)!} f^{(N-1)}(\tilde{x}_{JK})(x-\tilde{x}_{JK})^{N-1} \end{aligned}\right\} \text{poly of degree } N-1$$

$$+ R_N(x)$$

$$\therefore \int f(x)\psi_{JK}(x)\,dx = \cancel{\int f(x) R_N(x)}$$

$$= \int R_N(x)\,\psi_{JK}(x)\,dx$$



$$R_N(x) = \frac{1}{N!} f^{(N)}(\xi_{jK})(x-\tilde{x}_{JK})^N$$

$$|R_N(x)| \le \frac{1}{N!}\|f^{(N)}\|_\infty a^N (2^{-j-1})^N$$

$$= \text{const.}\ 2^{-N(j+1)}$$

$$|\langle f, \psi_{jk}\rangle| = \left|\int R_N(x)\psi_{jk}(x)\,dx\right|$$

$$\leq \text{const } 2^{-N(j+1)} \underbrace{\int |\psi_{jk}(x)|\,dx}_{\text{apply Cauchy-Schwarz}}$$

$$\leq \text{const } 2^{-N(j+1)} \cdot$$

$$\underbrace{|\text{support } \psi_{jk}|^{1/2}}_{a^{1/2} \cdot 2^{-j/2}} \cdot \underbrace{\left(\int |\psi_{jk}|^2\right)^{1/2}}_{=1}$$

$$\left[\ \left|\int_I f(x)\,dx\right| \leq \left(\int_I |f|^2\right)^{1/2} \left(\int_I 1^2\right)^{1/2} = \left(\int_I |f|^2\right)^{1/2} |I|^{1/2}\ \right]$$

$$\leq \text{const. } 2^{-N(j+1)} 2^{-j/2}$$

$$\leq \text{const } 2^{-Nj} 2^{-j/2}$$

Relation to approximation of smooth functions.

Theorem. Given $N \in \mathbf{N}$, assume that the function $f \in C^N(\mathbf{R})$, and that $f^{(N)} \in L^\infty(\mathbf{R})$. Assume that the function $\psi(x)$ has compact support, that $\int_{\mathbf{R}} x^m \psi(x)\, dx = 0$, for $0 \le m \le N-1$ and that $\int_{\mathbf{R}} |\psi_{j,k}(x)|^2\, dx = 1$ for all $j, k \in \mathbf{Z}$. Then there is a constant $C > 0$ depending only on N and $f(x)$ such that for every $j, k \in \mathbf{Z}$,

$$|\langle f, \psi_{j,k}\rangle| \le C 2^{-jN} 2^{-j/2}. = C\, 2^{-j(N+1/2)}$$

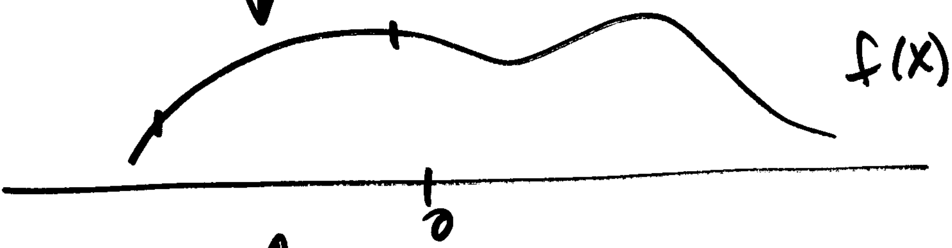
Reproduction of polynomials.

Theorem. Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA, and let $\psi(x)$ be the wavelet. If $\psi(x)$ has N vanishing moments, then for each integer $0 \le k \le N-1$, there are coefficients $\{q_{k,n}\}_{n \in \mathbf{Z}}$ such that

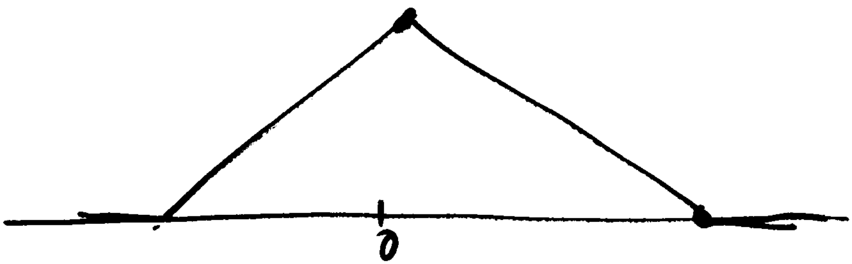
$$\sum_n q_{k,n}\, \varphi(x+n) = x^k.$$

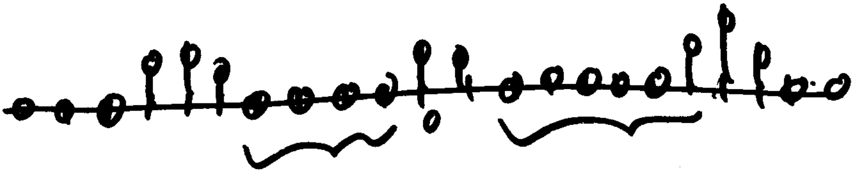
This means that every polynomial of degree $N-1$ can be written as a sum of shifted scaling ftns. In a sense it says that "degree $N-1$ polys are in V_0".

poly of degree N-1



all wavelet coeffs on this part of f are zero.





wavelet coeffs for large j.