10-30-03

MRA

wavelet recipe

φ – scaling ftn

h – scaling filter (2 scale dilation eqn)

g – wavelet filter $g(n) = (-1)^n \overline{h(1-n)}$

ψ – wavelet

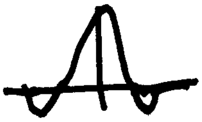
Gives you $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}} =$

$= \{D_{2^j} T_k \psi(x)\}_{j,k\in\mathbb{Z}}$ is an ONB on $\mathbb{R}$.

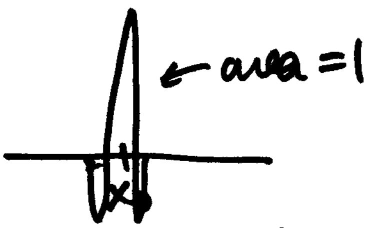
Motivation

(a) Suppose we have some samples of a signal $f(t)$, at $k\tau$, $k \in \mathbb{Z}$, $\tau > 0$ (usually small). Say $\tau = 2^{-J}$ some large $J > 0$.

Use the following principle of approx. delta

If a ftn g satisfies $\int g = 1$ g: 

Then $f(x_0) \approx \int f(t)\, g\!\left(\frac{t - x_0}{L}\right) L^{-1}\, dt$.



Lets take g to be the scaling ftn φ

know $|\int \varphi| = 1$ so we can assume that $\int \varphi = 1$ by normalizing

Then

$$\underline{f(k\tau)} = f(k2^{-J}) \approx \int f(t)\, \varphi\!\left(\frac{t - k2^{-J}}{2^{-J}}\right) 2^{J}\, dt$$

$$= \int f(t)\, 2^{J} \varphi(2^{J} t - k)\, dt$$

$$= 2^{J/2} \langle f, \varphi_{J,k} \rangle$$

Since $\langle f, \varphi_{J,k} \rangle = \underline{\langle D_{2^{-J}} f, \varphi_{0k} \rangle}$

(c) $\varphi_{jk} = D_{2^j} T_k \varphi = D_{2^j} T_k \left(\sum_n h(n) \varphi_{1,n} \right)$

$$= D_{2^j} T_k \left(\sum_n h(n) D_2 T_n \varphi \right)$$

$$= \sum_n h(n) D_{2^j} T_k D_2 T_n \varphi$$

$$= \sum_n h(n) D_{2^j} D_2 T_{2k} T_n \varphi$$

$$= \sum_n h(n) D_{2^{j+1}} T_{n+2k} \varphi \qquad n \mapsto n-2k$$

$$= \sum_n h(n-2k) \varphi_{j+1,n}$$

(d) $c_{j+1}(k) = \langle f, \varphi_{-j-1,k} \rangle$

$$= \langle f, \sum_n h(n-2k) \varphi_{-j,n} \rangle$$

$$= \sum_n \overline{h(n-2k)} \langle f, \varphi_{-j,n} \rangle$$

$$= \sum_n c_j(n) \overline{h(n-2k)}$$

$$d_{j+1}(k) = \sum_n c_j(n) \overline{g(n-2k)}$$

(e) Follows from identity

$$P_{-j} f = P_{-j-1} f + Q_{-j-1} f$$

$$\sum_n \langle f, \varphi_{-jn} \rangle \varphi_{-jn} = \sum_n \langle f, \varphi_{-j-1,n} \rangle \varphi_{-j-1,n} + \sum_n \langle f, \psi_{-j-1,n} \rangle \psi_{-j-1,n}$$

$$\sum_n c_j(n) \varphi_{-jn} = \sum_n c_{j+1}(n) \varphi_{-j-1,n} + \sum_n d_{j+1}(n) \psi_{-j-1,n}$$

$$= \sum_n c_{j+1}(n) \sum_k h(k-2n) \varphi_{-j,k} + \sum_n d_{j+1}(n) \sum_k g(k-2n) \varphi_{-jk}$$

$$= \sum_k \Big(\sum_n c_{j+1}(n) h(k-2n) + \sum_n d_{j+1}(n) g(k-2n) \Big) \varphi_{-jk}$$

Match coefficients

$$c_j(k) = \sum_n c_{j+1}(n) h(k-2n) + d_{j+1}(n) g(k-2n)$$

Pf: (a) Rcl $\hat{\varphi}(\gamma) = m_0(\gamma/2)\hat{\varphi}(\gamma/2)$

$$m_0(\gamma) = \frac{1}{\sqrt{2}}\sum_n h(n)\, e^{-2\pi i n\gamma}$$

$$\int \varphi \neq 0 \iff \hat{\varphi}(0) \neq 0$$

$$\therefore\ \hat{\varphi}(0) = m_0(0)\hat{\varphi}(0)$$

$$\therefore\ m_0(0) = 1 \iff \frac{1}{\sqrt{2}}\sum_n h(n) = 1$$

(b) $\hat{\psi}(\gamma) = m_1(\gamma/2)\,\hat{\varphi}(\gamma/2)$

$$m_1(\gamma) = e^{-2\pi i(\gamma+1/2)}\,\overline{m_0(\gamma+1/2)}$$

gives $m_1(0) = 0$.

(c) $$\langle \varphi_{00}, \varphi_{0n}\rangle = \left\langle \sum_m h(m)\varphi_{1,m},\ \sum_K h(K-2n)\varphi_{1,K}\right\rangle$$

$$= \sum_m \sum_K h(m)\overline{h(K-2n)}\underbrace{\langle \varphi_{1,m}, \varphi_{1,K}\rangle}_{\delta(m-K)}$$

$$= \sum_m h(m)\overline{h(m-2n)}$$

(d) same

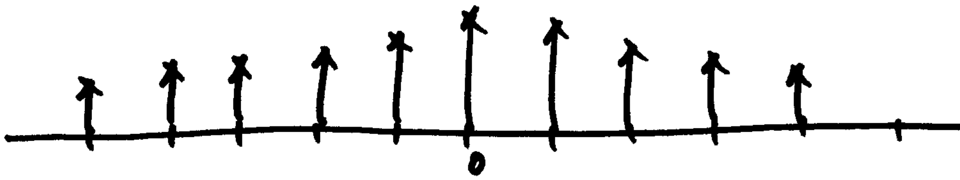
(e) know: $P_{j+1} = P_j + Q_j$ translates into

$$c_0(n) = \sum_K c_1(K)\,h(n-2K) + \sum_K d_1(K)\,g(n-2K)$$

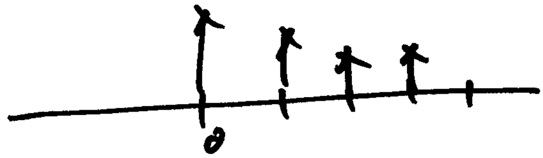
$$= \sum_K \sum_m c_0(m)\overline{h(m-2K)}h(n-2K) + c_0(m)\overline{g(m-2K)}g(n-2K)$$

$$= \sum_m c_0(m)\underbrace{\left(\sum_K \overline{h(m-2K)}h(n-2K) + \overline{g(m-2K)}g(n-2K)\right)}_{= 1 \text{ if } m=n;\ 0 \text{ if } m\neq n}$$

H :

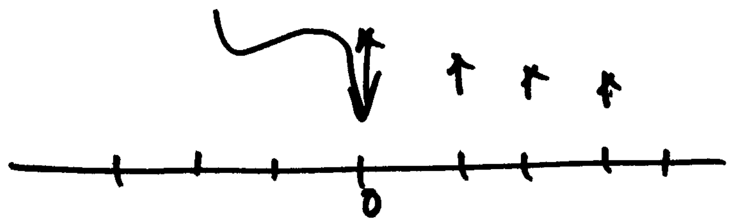


$c(n)$

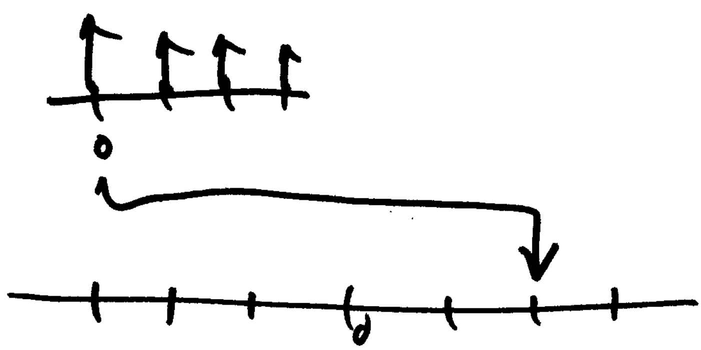


$h(n)$

$\underline{k=0}$

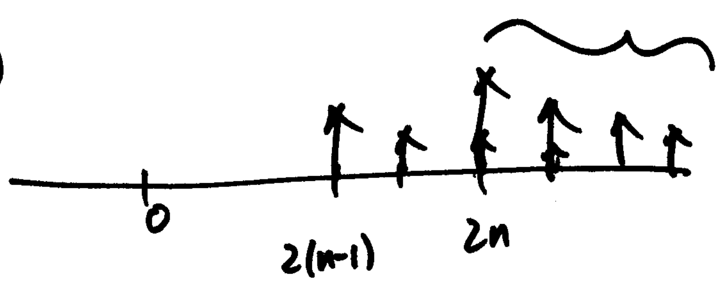


$\underline{k=1}$



mult by $c(n)$

$\underline{H^*}$! $h(k-2n)$



$$\sum_n c(n)h(k-2n)$$

H^*c is a weighted sum of shifts of h.

Now write: $c_{j+1} = Hc_j \quad d_{j+1} = Gc_j$

<u>and</u> $c_j = H^* c_{j+1} + G^* d_{j+1}$

Pf (a) Let $c(n)$ be given.

$$(HH^*c)(k) = \sum_n (H^*c)(n)\overline{h(n-2k)}$$

$$= \sum_n \sum_m c(m) h(n-2m)\overline{h(n-2k)}$$

$$= \sum_m c(m) \sum_n h(n-2m)\overline{h(n-2k)}$$

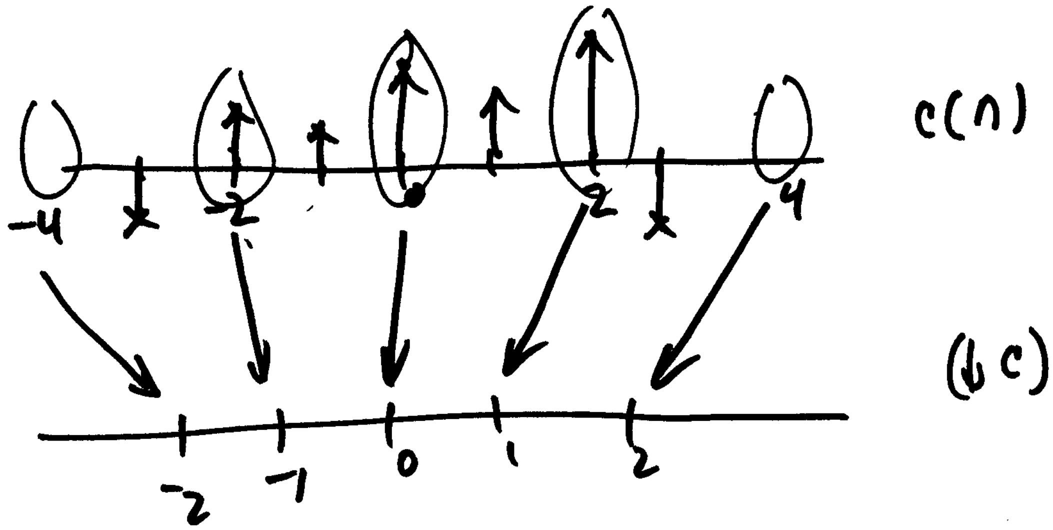
$$= c(k) \iff \sum_n h(n-2m)\overline{h(n-2k)} = \begin{cases} 1 & m=k \\ 0 & m\neq k \end{cases}$$

$$\iff \sum_n h(n) h(n-2(k-m)) = \begin{cases} 1 & m=k \\ 0 & m\neq k \end{cases}$$

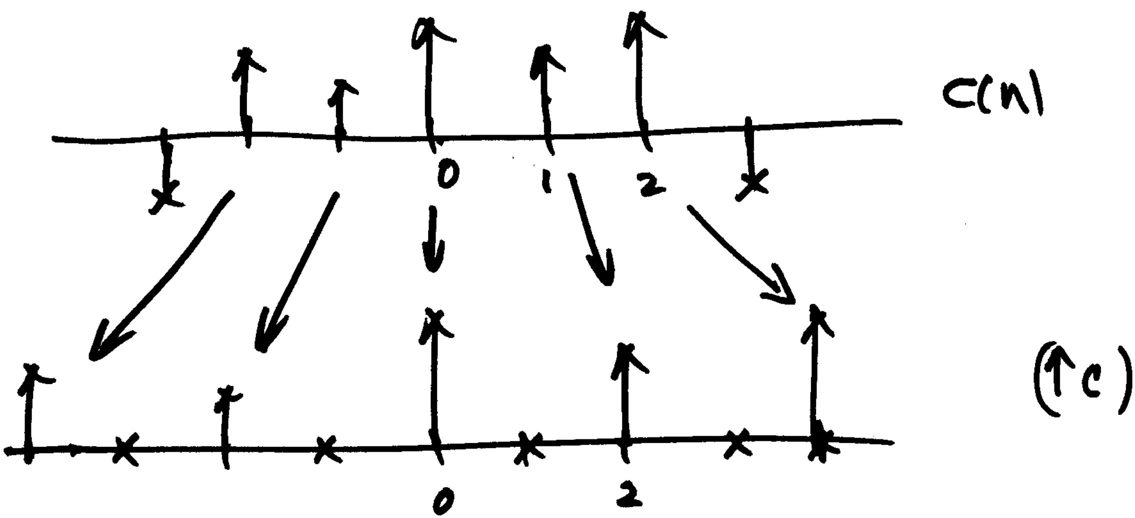
$$\iff \sum_n h(n) h(n-2l) = \begin{cases} 1 & l=0 \\ 0 & l\neq 0 \end{cases}.$$

(D) downsampling:

$$(\downarrow c)(n) = c(2n)$$

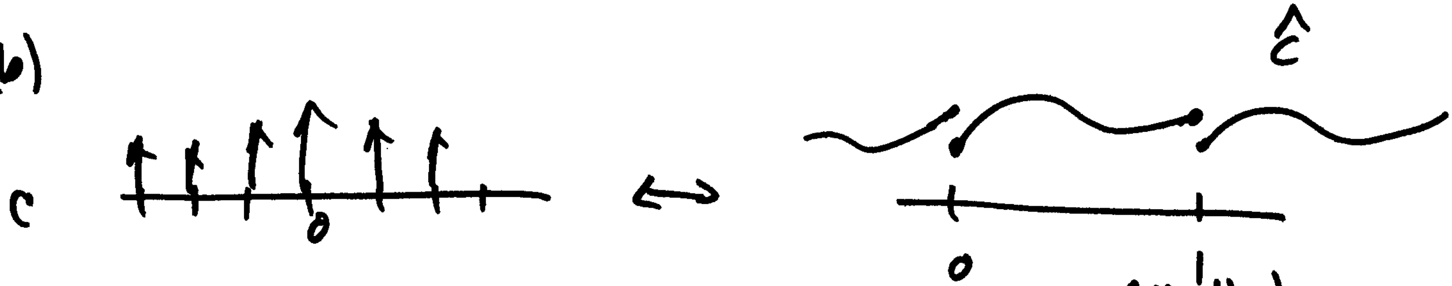


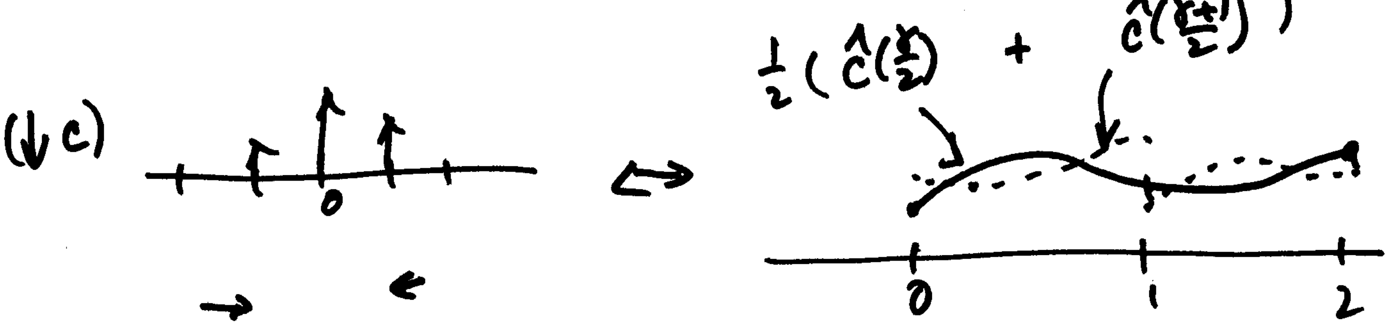
(r)



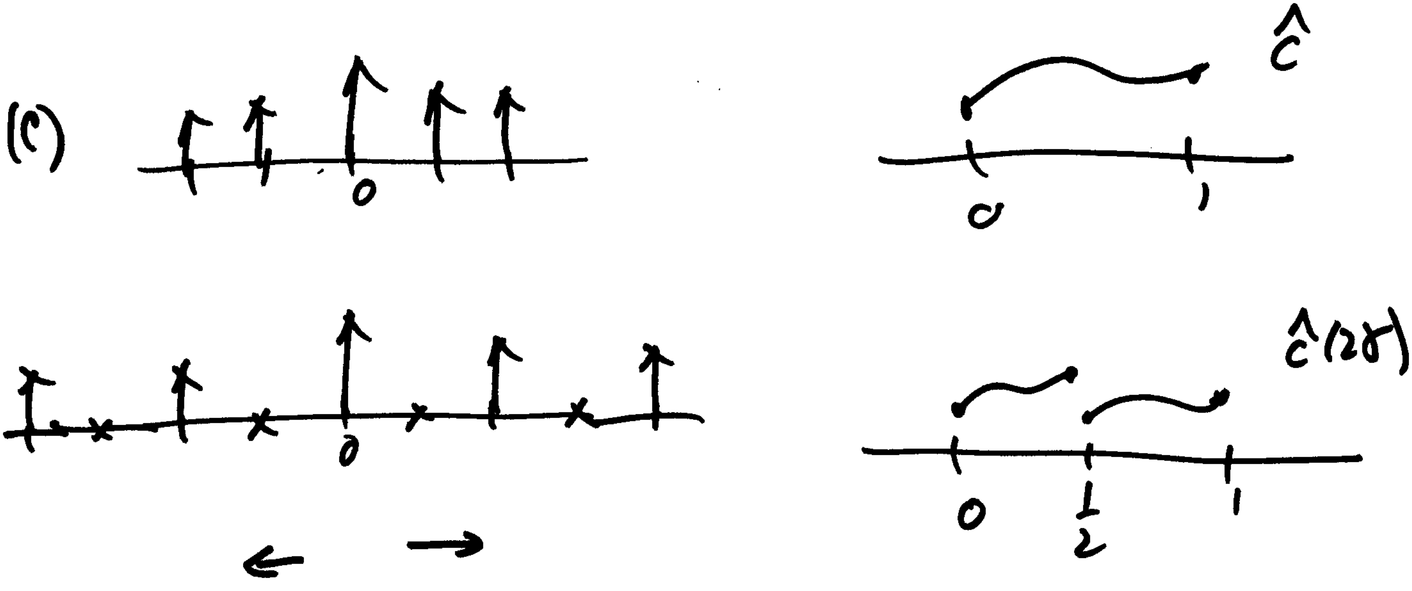
Lemma: (a) obvious

(b)





(c)



Pf: (a) $(c * \underline{h})(n) = \sum_m c(m)\, \underline{h}(n-m)$

$$= \sum_m c(m)\, \overline{h(m-n)}$$

$$\downarrow (c * \underline{h})(n) = \cancel{\sum_m} (c * \underline{h})(2n)$$

$$= \sum_m c(m)\, \overline{h(m-2n)} = Hc(n)$$

(b) $(\uparrow c) * h(n) = \sum_m (\uparrow c)(m)\, h(n-m)$

$$= \sum_{\substack{m \text{ even} \\ \text{or} \\ m = 2k}} c(k)\, h(n-2k)$$

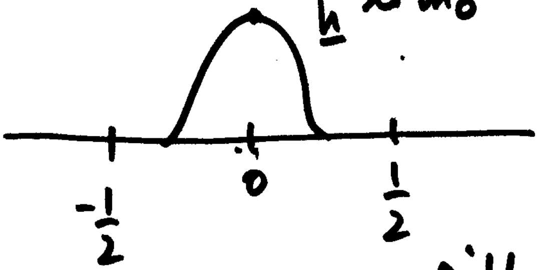
$$= \sum_k c(k)\, h(n-2k)$$

H: $m_0(\gamma) = \frac{1}{\sqrt{2}}\hat{\underline{h}}(\gamma)$

Since $\sum h(n) = \sqrt{2}$, $\hat{\underline{h}}(0) = 1$

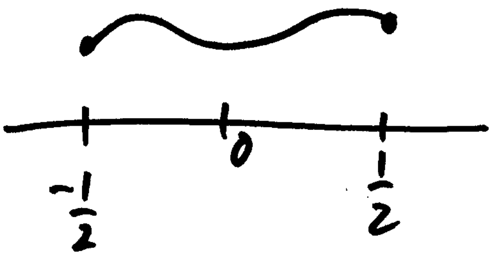
$\hat{\underline{h}} \sim m_0$

$\underline{m_0(\tfrac{1}{2}) = 0}$

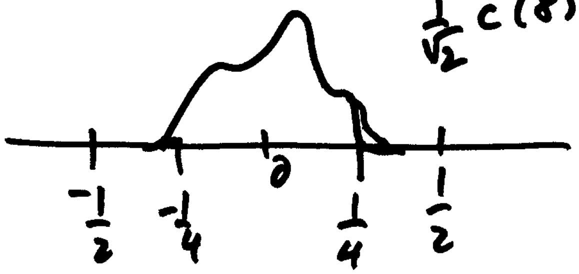


m_0 is a low-pass filter

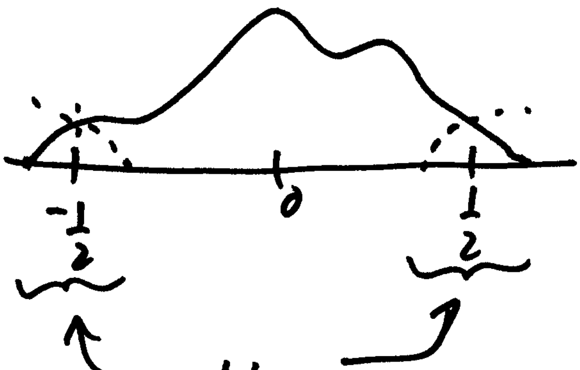
$c(n)$: $\hat{c}(\gamma)$



$c * \underline{h}$: $\frac{1}{\sqrt{2}}\hat{c}(\gamma)\overline{m_0(\gamma)}$



$\downarrow(c * \underline{h})$:

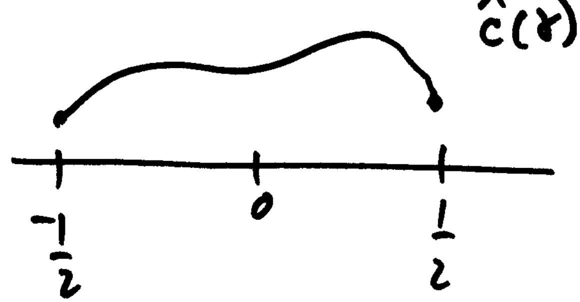


aliasing error (maybe not too bad)

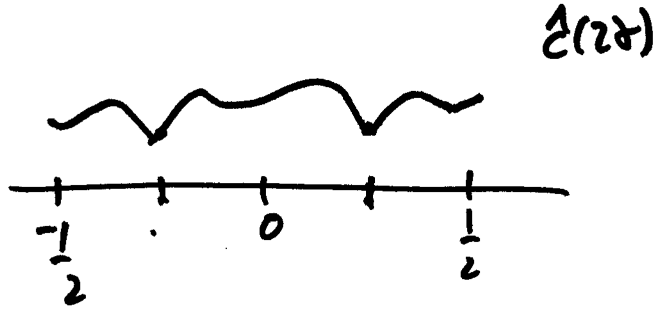
Hc captures the "low frequency part" of c.

Gc captures the "high frequency part" of c.

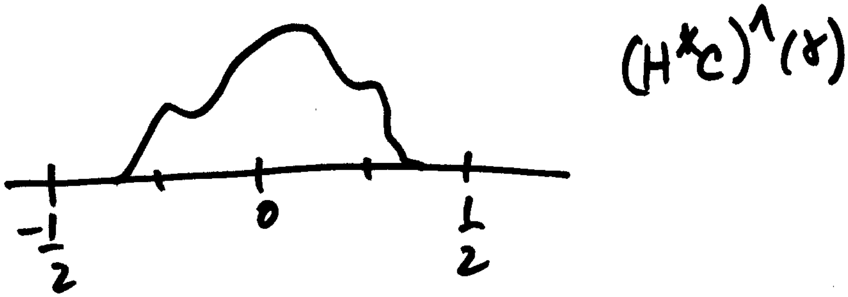
c :



$(\uparrow c)$



$(\uparrow c) * h$



$$
\begin{aligned}
(H^*Hc)^\wedge(\gamma) &= \sqrt{2}\ \widehat{Hc}(2\gamma)\, m_0(\gamma) \\
&= \left(\hat{c}(\gamma)\overline{m_0(\gamma)} + \hat{c}(\gamma+\tfrac{1}{2})\overline{m_0(\gamma+\tfrac{1}{2})}\right) m_0(\gamma) \\
&= \hat{c}(\gamma)|m_0(\gamma)|^2 + \underbrace{\hat{c}(\gamma+\tfrac{1}{2})\, m_0(\gamma)\overline{m_0(\gamma+1/2)}}_{\text{aliasing error}}
\end{aligned}
$$

Point: G and G^* have the property that $H^*Hc + \underline{G^*Gc} = c$

Lemma: Proof follows from definition of $m_1(\gamma)$. In fact we have seen this

Thm: Proof (a) $\Longleftrightarrow$ (b)

Seen: $(H^*Hc)^\wedge(\gamma) = \hat{c}(\gamma)|m_0(\gamma)|^2 + \hat{c}(\gamma+\tfrac{1}{2})m_0(\gamma)\overline{m_0(\gamma+\tfrac{1}{2})}$

$$(G^*Gc)^\wedge(\gamma) = \hat{c}(\gamma)\,|m_1(\gamma)|^2 + \hat{c}(\gamma+1/2)\,m_1(\gamma)\,\overline{m_1(\gamma+1/2)}$$

$$= \hat{c}(\gamma)\,|m_0(\gamma+1/2)|^2 + \hat{c}(\gamma+\tfrac{1}{2})\,m_1(\gamma)\,\overline{m_1(\gamma+1/2)}$$

$\therefore [(H^*H + G^*G)c]^\wedge(\gamma)$

$$= \hat{c}(\gamma)\left(|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2\right) + \hat{c}(\gamma+1/2)\underbrace{\left(m_0(\gamma)\overline{m_0(\gamma+1/2)} + m_1(\gamma)\overline{m_1(\gamma+1/2)}\right)}_{=0}$$

$$= \hat{c}(\gamma)\left(|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2\right)$$

$$= \hat{c}(\gamma) \quad \text{all } c(n)$$

$$\Longleftrightarrow |m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2 = 1$$

Definition. Given $h(k)$, $m_0(\gamma)$ as before, we say $h(k)$ is a QMF (quadrature mirror filter) if

(a) $m_0(0) = 1$ and $\Leftrightarrow \sum h(n) = \sqrt{2}$

(b) $|m_0(\gamma/2)|^2 + |m_0(\gamma/2 + 1/2)|^2 \equiv 1.$

Theorem. Suppose that $h(k)$ is a QMF. Define $g(k)$ as before. Then:

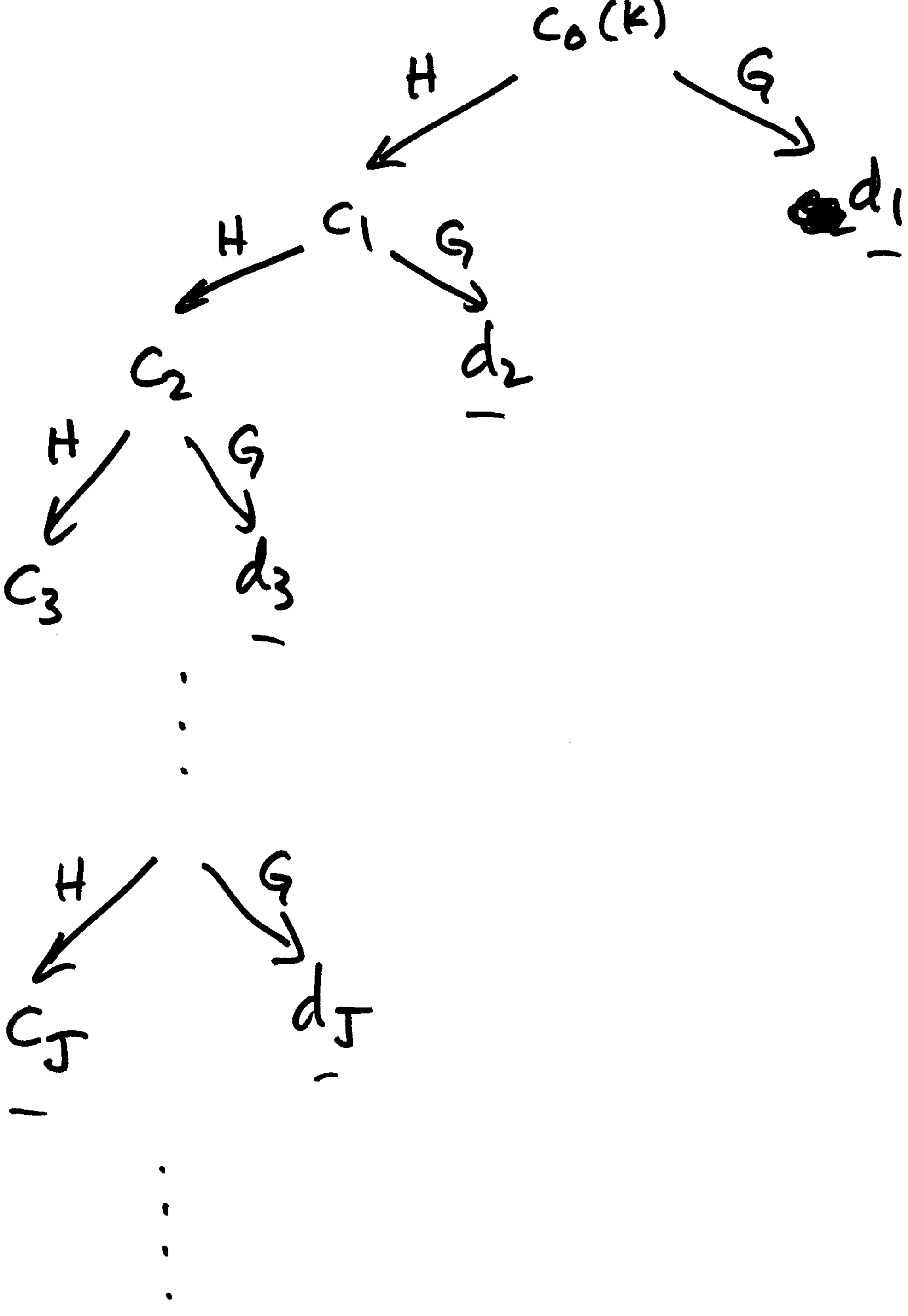
(a) $\sum_n h(n) = \sqrt{2}$,

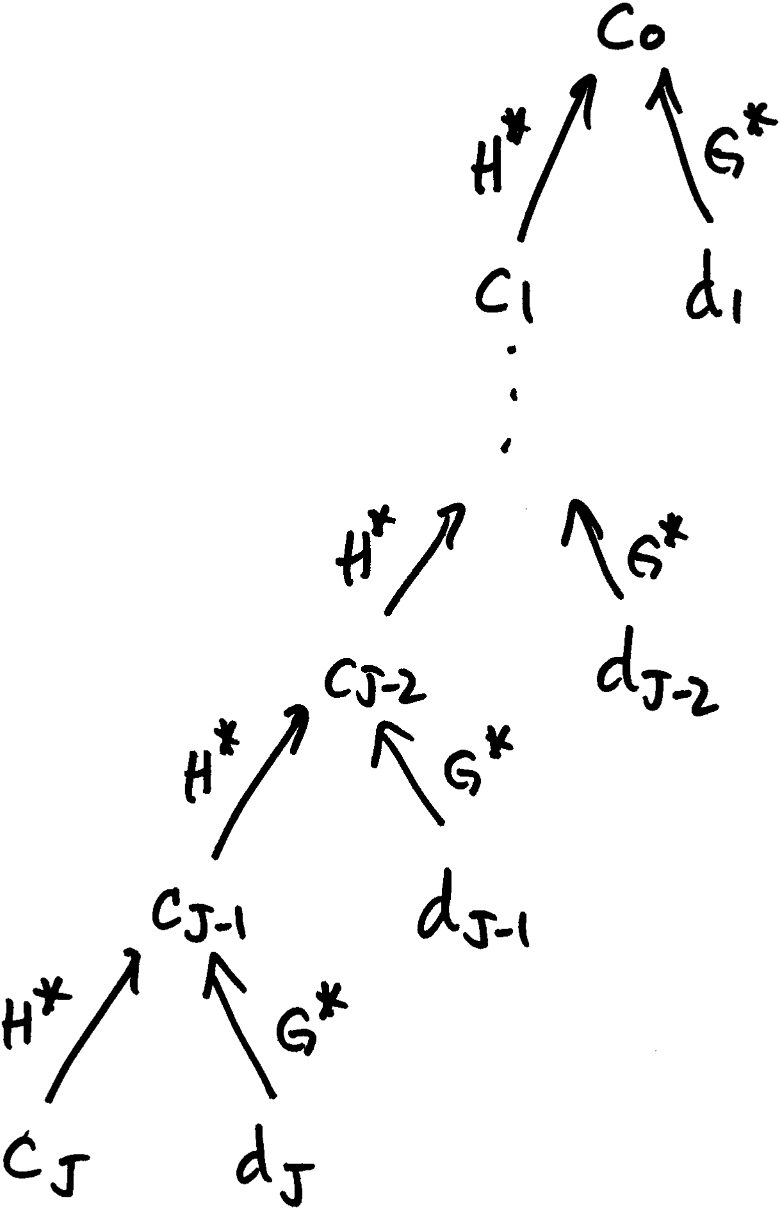
(b) $\sum_n g(n) = 0$,

(c) $\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n)$.

(d) $\sum_k g(k)\,\overline{h(k-2n)} = 0$ for all $n \in \mathbf{Z}$.

(e)
$$\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(n-m).$$


$c_0(k)$
H
G
c_1
$\underline{d_1}$
H
G
c_2
$\underline{d_2}$
H
G
c_3
$\underline{d_3}$
H
G
$\underline{c_J}$
$\underline{d_J}$


c_0
H^*
G^*
c_1
d_1
H^*
G^*
c_{J-2}
d_{J-2}
H^*
G^*
c_{J-1}
d_{J-1}
H^*
G^*
c_J
d_J