10-23-03

MRA

Examples

Wavelet "recipe"

$\varphi$ scaling ftn

$h(n)$ scaling filter : $\varphi = \sum_n h(n)\varphi_{1,n}$

$g(n) = (-1)^n \overline{h(1-n)}$

$\psi = \sum_n g(n)\varphi_{1,n}$

Thm: $\psi$ constructed in this way satisfies $\{\psi_{j,k}\}$ is an o.n. basis on $\mathbb{R}$.

---

* Will prove this theorem.

Pf: orthonormality:

Show $\langle \varphi_{jk}, \varphi_{jm} \rangle = \delta(k-m)$

$$\langle \varphi_{jk}, \varphi_{jm} \rangle = \langle D_{2^j} T_k \varphi, D_{2^j} T_m \varphi \rangle$$

$$= \langle T_k \varphi, T_m \varphi \rangle = \delta(k-m)$$

Completeness: Let $f \in V_j$

$$f \in V_j \Leftrightarrow D_{2^{-j}} f \in V_0 \Leftrightarrow D_{2^{-j}} f = \sum_k \langle D_{2^{-j}} f, \varphi_{0,k} \rangle \varphi_{0,k}$$

$$\Leftrightarrow f = D_{2^j} \sum_k \langle f, D_{2^j} \varphi_{0k} \rangle \varphi_{0k}$$

$$= \sum_k \langle f, \varphi_{jk} \rangle D_{2^j} \varphi_{0k} = \sum_k \langle f, \varphi_{jk} \rangle \varphi_{jk}$$

---

Pf: (a) Let $\varepsilon > 0$. Since $\overline{\text{span}\{V_j\}} = L^2$ there is a $g \in V_{j_0}$ such that $\|f - g\|_2 < \frac{\varepsilon}{2}$

If $j \geq j_0$ then $P_j g = g$ and

$$\|P_j f - f\|_2 = \|P_j f - P_j g + P_j g - f\|_2$$

$$\leq \|P_j f - P_j g\|_2 + \|g - f\|_2$$

$$= \|P_j (f-g)\|_2 + \|g - f\|_2$$

$$\leq \|f - g\|_2 + \|f - g\|_2 < \varepsilon.$$

[ $P_j h = \sum_k \langle h, \varphi_{jk} \rangle \varphi_{jk}$

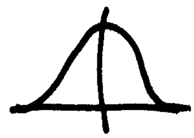
$\|P_j h\|_2^2 = \sum |\langle h, \varphi_{jk} \rangle|^2 \leq \|h\|_2^2$ by Bessel's $\leq$ ]

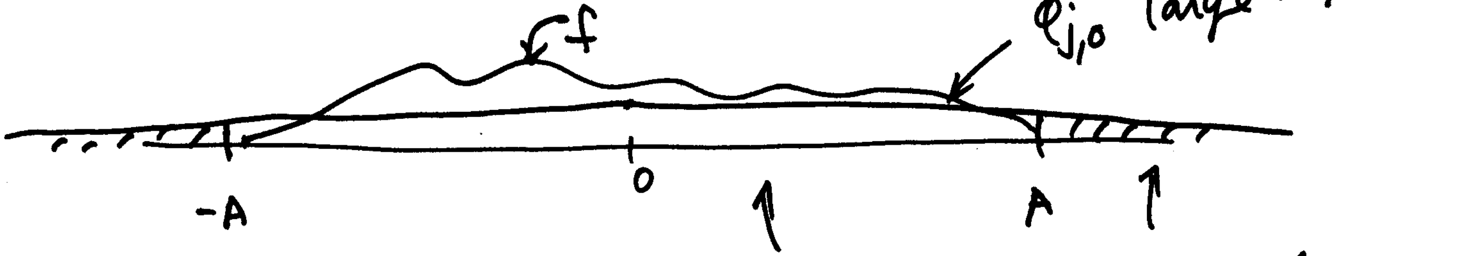
$$\therefore \lim_{j \to \infty} \|P_j f - f\|_2 = 0$$

(b) Show $\lim_{j \to -\infty} \|P_j f\|_2 = 0$ for $f \in C_c^0(\mathbb{R})$

Note: $\|P_j f\|_2^2 = \sum_k |\langle f, \varphi_{jk} \rangle|^2$

$\varphi(x)$






Specifically,

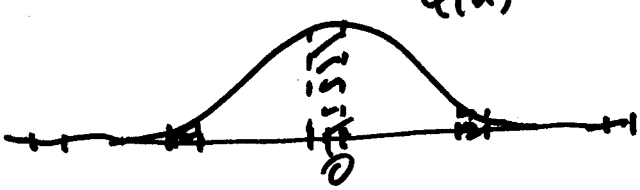
$$|\langle f, \varphi_{jk} \rangle|^2 = \left| \int_{-A}^{A} f(x)\, 2^{j/2} \varphi(2^j x - k)\, dx \right|^2$$

Cauchy Schwarz → $\leq \left( \int_{-A}^{A} |f(x)|^2 dx \right) \left( \int_{-A}^{A} 2^j\, |\varphi(2^j x - k)|^2 dx \right)$ ↖ $u = 2^j x - k$, $du = 2^j dx$

$$2^j \int_{-A}^{A} |\varphi(2^j x - k)|^2 dx = \int_{-2^j A - k}^{2^j A - k} |\varphi(u)|^2 du$$

$$\sum |\langle f, \varphi_{jk} \rangle|^2 \leq \|f\|_2^2 \sum_k \int_{[-2^j A, 2^j A] - k} |\varphi(u)|^2 du \longrightarrow 0 \quad \text{as } j \to -\infty$$

$\varphi(u)$



Remark: (a) says $W_0 = \text{span}\{T_n\psi\}$

+ $\{T_n\psi\}$ is an ONB for $W_0$.

(b) says $V_0 \perp W_0$

(c) says $V_1 = V_0 \oplus W_0$ (i.e. an orthogonal splitting)

Why? $f \in V_1$ then $P_1 f = f$ and by definition

$$P_1 f = P_0 f + Q_0 f$$

$P_1 f = f$, $P_0 f \in V_0$, $Q_0 f \in W_0$ by (c)

[Proof of Lemma is really verifying that you can replace 0 by j and everything works.]

---

Proof of Lemma:

(1) Show $\{\psi_{jk}\}$ is orthonormal system.

(a) within a scale j.

show $\langle \psi_{jk}, \psi_{jm} \rangle = \delta(k-m)$

$$\langle \psi_{jk}, \psi_{jm} \rangle = \langle D_2^j \psi_{0k}, D_2^j \psi_{0m} \rangle = \langle \psi_{0k}, \psi_{0m} \rangle$$
$$= \delta(k-m) \text{ by (a).}$$

(b) between scales.

$\langle \psi_{j,k}, \psi_{j',k'} \rangle$. $j \neq j'$ Assume $j' \geq j$

This means $V_j \subseteq V_{j'}$

Since $\psi \in V_1$, $\psi_{jk} \in V_{j+1} \subseteq V_{j'}$

$\therefore \psi_{jk} =$ ~~$\sum_m \langle \psi_{jk}, \varphi_{j+1,m} \rangle \varphi_{j+1,m}$~~

$$= \sum_m \underbrace{\langle \psi_{jk}, \varphi_{j',m} \rangle}_{(m)} \varphi_{j',m}$$

$$\therefore \langle \psi_{jk}, \psi_{j',k'} \rangle = \left\langle \sum_m \langle (m) \rangle \varphi_{j',m}, \psi_{j',k'} \right\rangle$$

$$= \sum_m \langle (m) \rangle \underbrace{\langle \varphi_{j',m}, \psi_{j',k'} \rangle}_{=0 \text{ all } m,k' \text{ by (b)}}$$

Why? $\langle \varphi_{j',m}, \psi_{j',k'} \rangle = \langle D_{2^{j'}} \varphi_{0,m}, D_{2^{j'}} \psi_{0,k'} \rangle$

$$= \langle \varphi_{0,m}, \psi_{0,k'} \rangle = 0 \text{ all } m, k'.$$

$\therefore \{\psi_{jk}\}$ is an o.n. system.

(2) Completeness:

Let $f$ be in $C_c^0(\mathbb{R})$, let $J > 0$ be large. Then

$$P_J f = (P_J f - P_{J-1} f) + (P_{J-1} f - P_{J-2} f) + (\cdots) + (P_{-J+1} f - P_{-J} f) + P_{-J} f$$

$$= Q_{J-1} f + Q_{J-2} f + \cdots + Q_{-J} f + P_{-J} f$$

$$= \sum_{j=-J}^{J-1} Q_j f + P_{-J} f.$$

Note that by (c) and (a), $Q_0 f = \sum_n \langle f, \psi_{0n} \rangle \psi_{0n}$

Also note that

$$Q_j f = P_{j+1} f - P_j f$$

$$= \sum_n \langle f, \varphi_{j+1,n} \rangle \varphi_{j+1,n} - \sum_n \langle f, \varphi_{jn} \rangle \varphi_{jn}$$

$$= D_{2^j} \Big( \overbrace{\sum_n \langle D_{2^{-j}} f, \varphi_{1,n} \rangle \varphi_{1,n}}^{P_1 D_{2^{-j}} f} - \overbrace{\sum_n \langle D_{2^{-j}} f, \varphi_{0n} \rangle \varphi_{0n}}^{P_0 D_{2^{-j}} f} \Big)$$

$$= D_{2^j} ( P_1 D_{2^{-j}} f - P_0 D_{2^{-j}} f )$$

$$= D_{2^j} (P_1 - P_0) D_{2^{-j}} f$$

$$= D_{2^j} Q_0 D_{2^{-j}} f .$$

$$\therefore Q_j f = D_{2^j} \Big( \sum_n \langle D_{2^{-j}} f, \psi_{0n} \rangle \psi_{0n} \Big)$$

$$= \sum_n \langle f, D_{2^j} \psi_{0n} \rangle D_{2^j} \psi_{0n}$$

$$= \sum_n \langle f, \psi_{jn} \rangle \psi_{jn} .$$

---

Going back ...

$$f \leftarrow P_J f - P_{-J} f = \sum_{j=-J}^{J-1} Q_j f = \sum_{j=-J}^{J-1} \sum_k \langle f, \psi_{jk} \rangle \psi_{jk}$$

Note that as $J \to \infty$, $\| f - P_J f + P_{-J} f \|_2 \le \| f - P_J f \|_2 + \| P_{-J} f \|_2$
$\to 0$.

$\therefore \{\psi_{jk}\}$ complete

Proof of (a): Since $\{\varphi_{0n}\}$ is an o.n. system,

$\langle \varphi_{00}, \varphi_{0n} \rangle = \delta(n)$. But from prev. calculation

$$\langle \varphi_{00}, \varphi_{0n} \rangle = \int_0^1 \sum_m |\hat{\varphi}(\gamma+m)|^2 e^{2\pi i n \gamma} d\gamma$$

Now, $\hat{\varphi}(\gamma) = m_0(\gamma/2)\, \hat{\varphi}(\gamma/2)$

$$= \int_0^1 \sum_m \left| m_0\left(\frac{\gamma+m}{2}\right) \hat{\varphi}\left(\frac{\gamma+m}{2}\right) \right|^2 e^{2\pi i n \gamma} d\gamma$$

$$= \int_0^1 \Big( \sum_{m=2k} |m_0(\tfrac{\gamma}{2}+k)|^2 \, |\hat{\varphi}(\tfrac{\gamma}{2}+k)|^2 + \sum_{m=2k+1} |m_0(\tfrac{\gamma}{2}+k+\tfrac{1}{2})|^2 \, |\hat{\varphi}(\tfrac{\gamma}{2}+k+\tfrac{1}{2})|^2 \Big) e^{2\pi i n \gamma} d\gamma$$

$$= \int_0^1 \Big( |m_0(\tfrac{\gamma}{2})|^2 \underbrace{\sum_k |\hat{\varphi}(\tfrac{\gamma}{2}+k)|^2}_{1} + |m_0(\tfrac{\gamma}{2}+\tfrac{1}{2})|^2 \underbrace{\sum_k |\hat{\varphi}(\tfrac{\gamma}{2}+\tfrac{1}{2}+k)|^2}_{1} \Big) e^{2\pi i n \gamma} d\gamma$$

Since $\{T_n \varphi\}$ is o.n. system, $\sum_k |\hat{\varphi}(t+k)|^2 \equiv 1$

$$= \int_0^1 \left( |m_0(\gamma/2)|^2 + |m_0(\gamma/2 + 1/2)|^2 \right) e^{2\pi i n \gamma} d\gamma.$$

So $\{T_n \varphi\}$ on system $\Longleftrightarrow$ $|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2 = 1$.

Show $\{T_n \psi\}$ is o.n. Look at $\sum_n |\hat{\psi}(\gamma+n)|^2$

Note: $\hat{\psi}(\gamma) = m_1(\gamma/2)\hat{\varphi}(\gamma/2)$

$$\sum_n |\hat{\psi}(\gamma+n)|^2 = \sum_n |m_1(\tfrac{\gamma+n}{2})|^2 |\hat{\varphi}(\tfrac{\gamma+n}{2})|^2$$

$$= |m_1(\gamma/2)|^2 \sum_k |\hat{\varphi}(\tfrac{\gamma}{2}+k)|^2 + |m_1(\gamma/2+1/2)|^2 \sum_k |\hat{\varphi}(\tfrac{\gamma}{2}+k+\tfrac{1}{2})|^2$$

$$= |m_1(\gamma/2)|^2 + |m_1(\gamma/2+1/2)|^2$$

[Recall $m_1(\gamma) = e^{-2\pi i(\gamma+1/2)}\overline{m_0(\gamma+1/2)}$]

$$= |m_0(\gamma/2+1/2)|^2 + |m_0(\gamma/2)|^2 = 1$$

$\therefore$ (a) is verified.

---

to prove (b):

$$m_1(\gamma/2)\overline{m_0(\gamma/2)} + m_1(\gamma/2+1/2)\overline{m_0(\gamma/2+1/2)}$$

$$= e^{-2\pi i(\gamma/2+1/2)}\overline{m_0(\gamma/2+1/2)}\,\overline{m_0(\gamma/2)} + e^{-2\pi i\gamma/2}\overline{m_0(\gamma/2)}\,\overline{m_0(\gamma/2+1/2)}$$

$$= -e^{-2\pi i\gamma/2}\overline{m_0(\gamma/2+1/2)}\,\overline{m_0(\gamma/2)} + e^{-2\pi i\gamma/2}\overline{m_0(\gamma/2)}\,\overline{m_0(\gamma/2+1/2)}$$

$$= 0$$

Now look at $\langle T_n \varphi, T_m \psi \rangle$

$$\langle T_n \varphi, T_m \psi \rangle = \int e^{-2\pi i n \gamma} \hat{\varphi}(\gamma) e^{2\pi i m \gamma} \overline{\hat{\psi}(\gamma)} \, d\gamma$$

$$= \int \hat{\varphi}(\gamma) \overline{\hat{\psi}(\gamma)} e^{2\pi i (m-n)\gamma} \, d\gamma$$

$$= \int_0^1 \sum_k \hat{\varphi}(\gamma+k) \overline{\hat{\psi}(\gamma+k)} \, e^{2\pi i (m-n)\gamma} \, d\gamma$$

$$= \int_0^1 \sum_k m_0\left(\frac{\gamma+k}{2}\right) \hat{\varphi}\left(\frac{\gamma+k}{2}\right) \overline{m_1\left(\frac{\gamma+k}{2}\right) \hat{\varphi}\left(\frac{\gamma+k}{2}\right)} \, e^{2\pi i (m-n)\gamma} \, d\gamma$$

$$= \int_0^1 \sum_k m_0\left(\frac{\gamma+k}{2}\right) \overline{m_1\left(\frac{\gamma+k}{2}\right)} \left|\hat{\varphi}\left(\frac{\gamma+k}{2}\right)\right|^2 e^{2\pi i (m-n)\gamma} \, d\gamma$$

$$= \int_0^1 \underbrace{\left(m_0\left(\frac{\gamma}{2}\right) \overline{m_1\left(\frac{\gamma}{2}\right)} + m_0\left(\frac{\gamma}{2}+\frac{1}{2}\right) m_1\left(\frac{\gamma}{2}+\frac{1}{2}\right)\right)} e^{2\pi i (m-n)\gamma} \, d\gamma$$

$$= 0$$

$\therefore$ (b) is verified.

(c) We have to verify that $Q_0 f \in \overline{\text{span}}\{T_n\psi\}$

$$Q_0 f = P_1 f - P_0 f \qquad P_1 f = \sum a(k)\varphi_{1,k}$$

$$P_0 f = \sum b(k)\varphi_{0,k}$$

$\widehat{P_1 f}(\gamma) = a(\gamma/2)\hat{\varphi}(\gamma/2)$ where $a(\gamma) = \frac{1}{\sqrt{2}}\sum a(k)e^{-2\pi i k\gamma}$

$\widehat{P_0 f}(\gamma) = b(\gamma)\hat{\varphi}(\gamma) = b(\gamma)m_0(\gamma/2)\hat{\varphi}(\gamma/2)$, $b(\gamma) = \sum b(n)e^{-2\pi i n\gamma}$

$Q_0 f \in \overline{\text{span}}\{T_n\psi\} \Leftrightarrow Q_0 f = \sum_k c(k)\psi_{0,k}$

$\Leftrightarrow \widehat{Q_0 f} = \hat{c}(\gamma)\hat{\psi}(\gamma) = \hat{c}(\gamma)m_1(\gamma/2)\hat{\varphi}(\gamma/2)$

where $\hat{c}(\gamma) = \sum c(k)e^{-2\pi i k\gamma}$

$\widehat{P_1 f} = \widehat{P_0 f} + \widehat{Q_0 f}$ becomes

$$a(\gamma/2)\hat{\varphi}(\gamma/2) = b(\gamma)m_0(\gamma/2)\hat{\varphi}(\gamma/2) + \hat{c}(\gamma)m_1(\gamma/2)\hat{\varphi}(\gamma/2)$$

Replace $\gamma$ by $\gamma+1$

$$a(\tfrac{\gamma}{2}+\tfrac{1}{2}) = b(\gamma)m_0(\tfrac{\gamma}{2}+\tfrac{1}{2}) + \hat{c}(\gamma)m_1(\tfrac{\gamma}{2}+\tfrac{1}{2})$$

This is the linear system

$$\underbrace{\begin{bmatrix} m_0(\gamma/2) & m_1(\gamma/2) \\ m_0(\gamma/2+1/2) & m_1(\gamma/2+1/2) \end{bmatrix}}_{M}\begin{bmatrix} b(\gamma) \\ \hat{c}(\gamma) \end{bmatrix} = \begin{bmatrix} a(\gamma/2) \\ a(\gamma/2+1/2) \end{bmatrix}$$

Can show $M^* M = I$

$$M^* M = \begin{bmatrix} \overline{m_0(\gamma/2)} & \overline{m_0(\gamma/2+1/2)} \\ \overline{m_1(\gamma/2)} & \overline{m_1(\gamma/2+1/2)} \end{bmatrix} \begin{bmatrix} m_0(\gamma/2) & m_1(\gamma/2) \\ m_0(\gamma/2+1/2) & m_1(\gamma/2+1/2) \end{bmatrix}$$

$$= \begin{bmatrix} |m_0(\gamma/2)|^2 + |m_0(\gamma/2+1/2)|^2 & \overline{m_0(\gamma/2)}\, m_1(\gamma/2) + \overline{m_0(\gamma/2+1/2)}\, m_1(\gamma/2+1/2) \\ m_0(\gamma/2)\overline{m_1(\gamma/2)} + m_0(\gamma/2+1/2)\overline{m_1(\gamma/2+1/2)} & |m_1(\gamma/2)|^2 + |m_1(\gamma/2+1/2)|^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $\hat{C}(\gamma)$ exists and $Q_0 f \in \overline{\operatorname{span}}\{T_n \psi\}$

$\therefore$ (c) is verified so $\{\psi_{jk}\}$ is an ONB on $\mathbb{R}$.

**Pf:** (a) Will show that $|\hat{\varphi}(0)| = 1$

$$\hat{\varphi}_{jk}(\gamma) = \widehat{D_{2^j} T_k \varphi}(\gamma) = D_{2^{-j}}(E_{-k}\hat{\varphi})(\gamma)$$
$$= 2^{-j/2} e^{-2\pi i k (2^{-j}\gamma)} \hat{\varphi}(2^{-j}\gamma)$$

Spse $\hat{f} \in C_c^0(\mathbb{R})$. Then $f \in L^2$ and

$$\|P_j f\|_2^2 = \sum_k |\langle f, \varphi_{jk}\rangle|^2 = \sum_k |\langle \hat{f}, \hat{\varphi}_{jk}\rangle|^2$$

$$\langle \hat{f}, \hat{\varphi}_{jk}\rangle = \int_{-\infty}^{\infty} \hat{f}(\gamma)\, 2^{-j/2} e^{+2\pi i k\gamma/2^j}\, \overline{\hat{\varphi}(2^{-j}\gamma)}\, d\gamma$$
$$= \int_{-2^{j-1}}^{2^{j-1}} \hat{f}(\gamma)\, \overline{\hat{\varphi}(2^{-j}\gamma)}\, \overline{\left(2^{-j/2} e^{2\pi i k\gamma/2^j}\right)}\, d\gamma$$

if $j$ large enough.

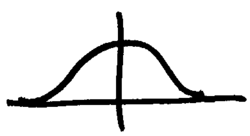
$= k^{th}$ Fourier coeff of $\hat{f}(\gamma)\,\overline{\hat{\varphi}(2^{-j}\gamma)}$

$$\sum_k |\langle \hat{f}, \hat{\varphi}_{jk}\rangle|^2 = \int_{-2^{j-1}}^{2^{j-1}} |\hat{f}(\gamma)|^2\, |\hat{\varphi}(2^{-j}\gamma)|^2\, d\gamma$$
$$= \int_{-A}^{A} |\hat{f}(\gamma)|^2\, |\hat{\varphi}(2^{-j}\gamma)|^2\, d\gamma \quad \text{let } j \to \infty.$$

$$|\hat{\varphi}(2^{-j}\gamma)|^2 \longrightarrow |\hat{\varphi}(0)|^2 \qquad \sum_k |\langle \hat{f}, \hat{\varphi}_{jk}\rangle|^2 \to \|f\|_2^2$$

$$\|f\|_2^2 = |\hat{\varphi}(0)|^2\, \|\hat{f}\|_2^2 = |\hat{\varphi}(0)|^2\, \|f\|_2^2$$

$\therefore |\hat{\varphi}(0)| = 1.$



(b) We will show $\hat{\psi}(0)=0$

$$\hat{\psi}(\gamma)=m_1(\gamma/2)\,\hat{\varphi}(\gamma/2) \Rightarrow \hat{\psi}(0)=m_1(0)\,\hat{\varphi}(0)$$

So show $m_1(0)=0$.

But $m_1(\gamma)=e^{-2\pi i(\gamma+1/2)}\,\overline{m_0(\gamma+1/2)}$

so $m_1(0)=-\overline{m_0(1/2)}$

So show $m_0(1/2)=0$

But since $\hat{\varphi}(\gamma)=m_0(\gamma/2)\,\hat{\varphi}(\gamma/2)$, $\hat{\varphi}(0)=m_0(0)\,\hat{\varphi}(0)$

so $m_0(0)=1$. Also

$$|m_0(\gamma)|^2+|m_0(\gamma+1/2)|^2=1$$

so $|m_0(0)|^2+|m_0(1/2)|^2=1$

so $1+|m_0(1/2)|^2=1$ $\quad\therefore m_0(1/2)=0$.

(c) Since $\{T_n\varphi\}$ is an on system, $\sum_n |\hat{\varphi}(\gamma+n)|^2=1$

Let $\gamma=0$ this is $\sum_n |\hat{\varphi}(n)|^2=1$. But we know $|\hat{\varphi}(0)|^2=1$

$\therefore \hat{\varphi}(n)=0$ for $n\neq 0$.

(d) $\sum \varphi(x+n)$ has period 1. Look at its Fourier coeffs

$$\int_0^1 \sum_n \varphi(x+n)\, e^{2\pi i n x}\,dx = \int_{-\infty}^{\infty} \varphi(x)\, e^{2\pi i n x}\,dx = \hat{\varphi}(n) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0. \end{cases}$$

$$\therefore \sum_n \varphi(x+n)=1 \text{ all } x.$$