10-16-03

MRA

Examples

(a) Haar MRA

(b) Band limited MRA

(c) Meyer MRA

(d) Piecewise linear MRA ← ⊛

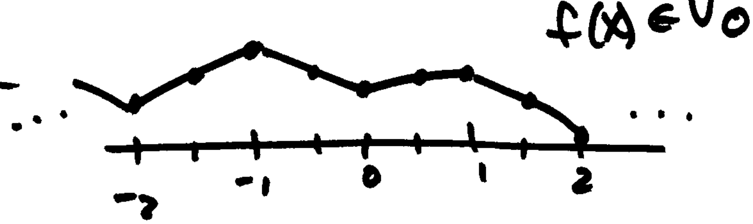
MRA: $\{V_j\}_{j\in\mathbb{Z}} \subseteq L^2$

1. $V_j \subseteq V_{j+1}$
2. $\overline{\operatorname{span}}\{V_j\} = L^2$
3. $\bigcap V_j = \{0\}$
4. $f \in V_j \iff D_{2^{-j}} f \in V_0$
5. $\exists\, \varphi$, scaling ftn, s.t. $\{T_n \varphi\}$ is ONB for V_0.

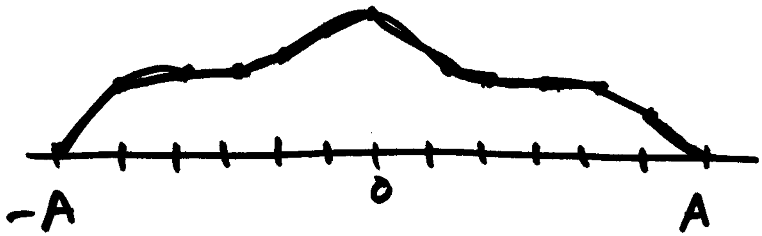
MRA completely determined by $\underline{\varphi}$ or by $\underline{V_0}$.

to verify MRA:

1. Nesting property



2. $\overline{\text{span}}\{V_j\} = L^2$

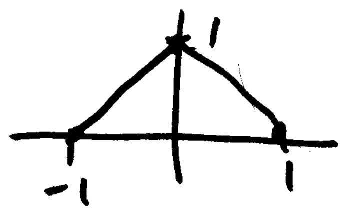


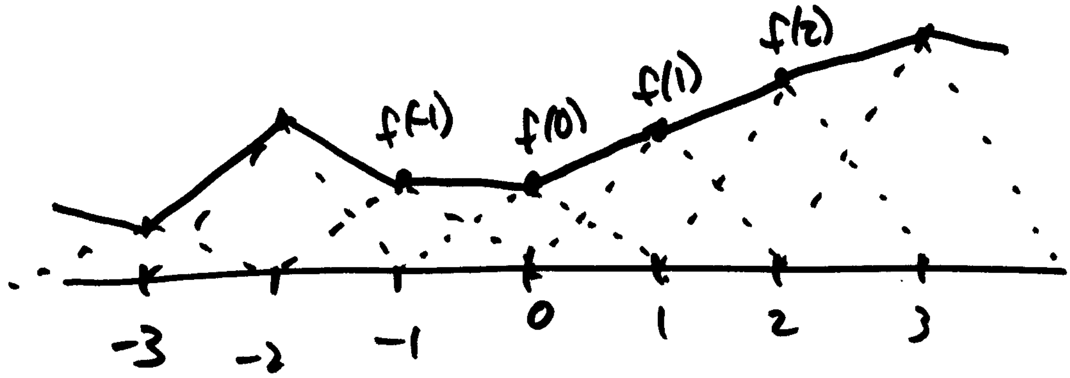
3. $\cap V_j = \{0\}$

4. $f \in V_j \Leftrightarrow D_{2^{-j}} f \in V_0$ By definition

5. Existence of φ requires some work.

$\varphi(x)$:





Any $f \in V_0$ has $f(x) = \sum_n f(n)\varphi(x-n)$ $\forall x$.

What about L^2 convergence?

(a) The function $\sum_n f(n)\varphi(x-n) \in L^2(\mathbb{R}) \iff$

$$\sum_n |f(n)|^2 < \infty$$

Note: $\{T_n\varphi\}$ is <u>not</u> orthonormal. Otherwise this would be trivial.

Spse $f(x)$ is linear on $[n, n+1)$.

Then $f(x) = f(n) + (f(n+1) - f(n))(x-n)$

$$\int_n^{n+1} |f(x)|^2 dx = \int_n^{n+1} |f(n) + (f(n+1)-f(n))(x-n)|^2 dx$$

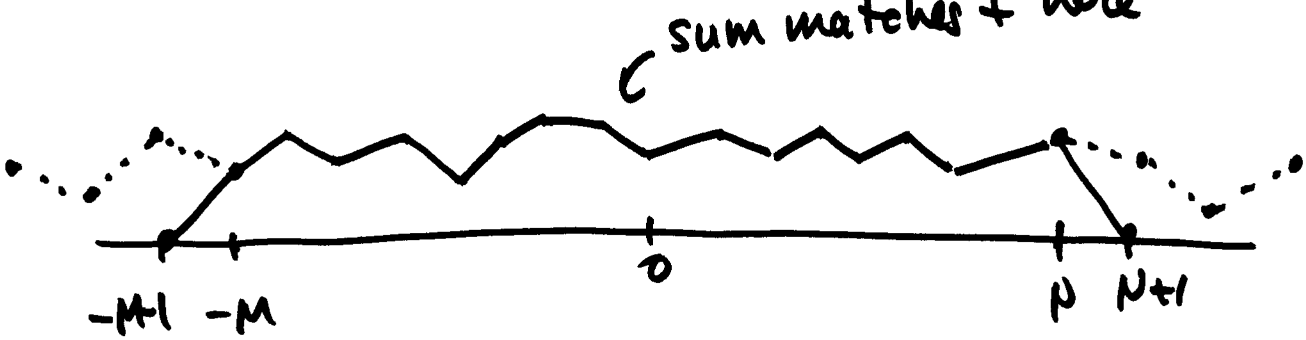
$$= \frac{1}{3}|f(n)|^2 + \frac{1}{3}|f(n+1)|^2 + \frac{1}{6}\left(\overline{f(n)}f(n+1) + f(n)\overline{f(n+1)}\right)$$

Can show that above is $\leq \frac{1}{2}(|f(n)|^2 + |f(n+1)|^2)$

$\geq \frac{1}{6}($ " $)$

Result follows.

(b) $\left\| f(x) - \sum_{n=-M}^{N} f(n)\varphi(x-n) \right\|_2 \longrightarrow 0$ as $M, N \to \infty$.



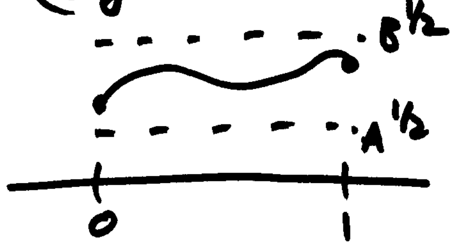
$$\therefore \text{Difference} = \begin{cases} 0 & -M \le x \le N \\ f(x) & -M-1 > x \text{ or } x > N+1 \\ f(-M)(x+M+1) & -M-1 \le x \le -M \\ f(N)(N+1-x) & N \le x \le N+1 \end{cases}$$

$$\|\text{Diff}\|_2^2 = \underbrace{\int_{[-M-1,N+1]^c} |f(x)|^2\,dx}_{\text{small since } f \in L^2} + \text{const}\underbrace{\left(|f(M)|^2 + |f(N)|^2\right)}_{\text{small since } |f(N)|^2 \to 0 \text{ as } N\to\infty}$$

<u>Pf</u>: (i) Define $\tilde{g}$ by

$$\hat{\tilde{g}}(\gamma) = \frac{\hat{g}(\gamma)}{\left(\sum_{n=-\infty}^{\infty} |\hat{g}(\gamma+n)|^2\right)^{1/2}}$$

← period 1 ftn. Bounded above (by $B^{1/2}$) and below (by $A^{1/2}$).



Then $\tilde{g} \in L^2$ because $\hat{g} \in L^2$. Also

$$\sum_{m=-\infty}^{\infty} |\hat{\tilde{g}}(\gamma+m)|^2 = \sum_{m=-\infty}^{\infty} \frac{|\hat{g}(\gamma+m)|^2}{\sum_n |\hat{g}(\gamma+m+n)|^2}$$

$$= \frac{\sum_m |\hat{g}(\gamma+m)|^2}{\sum_n |\hat{g}(\gamma+n)|^2} \equiv 1$$

$\therefore$ $\{T_n \tilde{g}\}_{n\in\mathbb{Z}}$ is an o.n. system.

(ii) Recall that $f \in \overline{\text{span}}\{T_n g\} \iff \hat{f}(\gamma) = c(\gamma)\hat{g}(\gamma)$ where $c(\gamma)$ has period 1.

$$\therefore \hat{\tilde{g}}(\gamma) = \hat{g}(\gamma) \cdot \frac{1}{\left(\sum_n |\hat{g}(\gamma+n)|^2\right)^{1/2}}$$

⇐ periodic

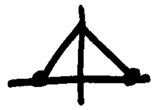
$\therefore$ $\tilde{g} \in \overline{\text{span}}\{T_n g\}$.

$$\hat{g}(\gamma) = \hat{\tilde{g}}(\gamma) \cdot \left(\sum_n |\hat{g}(\gamma+n)|^2\right)^{1/2}$$

$\therefore$ $g \in \overline{\text{span}}\{T_n \tilde{g}\}$.

This is sufficient for (ii).

<u>Pf</u>: Need to show that $\exists\ 0 < A < B$ s.t.

$$A \le \sum_n |\hat{\varphi}(\gamma+n)|^2 \le B \quad \forall \gamma. \qquad \varphi:$$ 

Recall a previous argument:

For <u>any</u> function g:

$$\langle T_n g, g\rangle = \int_{-\infty}^{\infty} T_n g(x)\,\overline{g(x)}\,dx = \int_{-\infty}^{\infty} e^{-2\pi i n t}\,|\hat{g}(t)|^2\,dt$$

$$= \int_0^1 e^{-2\pi i n t} \sum_n |\hat{g}(t+n)|^2\,dt$$

$= n^{th}$ Fourier coeff of $\sum_n |\hat{g}(t+n)|^2$.

So $$\sum_n |\hat{g}(\gamma+n)|^2 = \sum_m \langle T_m g, g\rangle e^{2\pi i m\gamma}$$

With $\varphi(x) = (1-|x|)\chi_{[-1,1]}(x)$,

$\langle \varphi, \varphi\rangle = \frac{2}{3}$, $\langle T_1\varphi, \varphi\rangle = \langle T_{-1}\varphi, \varphi\rangle = \frac{1}{6}$, $\langle T_m\varphi, \varphi\rangle = 0$ if $m \ne 0, 1, -1$.

$$\therefore \sum_n |\hat{\varphi}(\gamma+n)|^2 = \frac{1}{6}e^{-2\pi i\gamma} + \frac{2}{3} + \frac{1}{6}e^{2\pi i\gamma}$$

$$= \frac{2}{3} + \frac{1}{3}\cos(2\pi\gamma)$$

$$> \frac{1}{3} + \frac{1}{3}(1+\cos(2\pi\gamma))$$

$$= \frac{1}{3} + \frac{2}{3}\cos^2(\pi\gamma) = \frac{1}{3}(1+2\cos^2\pi\gamma)$$

$$\therefore \frac{1}{3} \le \sum |\hat{\varphi}(\gamma+n)|^2 \le 1.$$

$\underline{\text{Pf}}$: $\varphi(x) \in V_0 \subseteq V_1$ $\qquad$ $\varphi_{1,n}(x) = D_2 T_n \varphi = 2^{1/2}\varphi(2x-n)$

$\underline{\text{Note}}$: $\{\varphi_{1,n}\}_{n\in\mathbb{Z}}$ is an onb for V_1

(a) $\langle \varphi_{1,n}, \varphi_{1,m} \rangle = \langle D_2 T_n \varphi, D_2 T_m \varphi \rangle$

$= \langle T_n \varphi, T_m \varphi \rangle = \delta_{nm}$

$\therefore$ $\{\varphi_{1,n}\}$ an on system.

(b) $f \in V_1 \iff D_{2^{-1}} f \in V_0 \iff D_{2^{-1}} f = \sum_n \langle D_{2^{-1}} f, T_n \varphi \rangle T_n \varphi$

$\iff D_{2^{-1}} f = \sum_n \langle f, D_2 T_n \varphi \rangle T_n \varphi$

$\iff f = D_2 \left(\sum_n \langle f, D_2 T_n \varphi \rangle T_n \varphi \right)$

$= \sum_n \langle f, D_2 T_n \varphi \rangle D_2 T_n \varphi$

$= \sum_n \langle f, \varphi_{1,n} \rangle \varphi_{1,n}$ $\qquad$ $\therefore V_1 = \overline{\text{span}}\{\varphi_{1,n}\}$.

$\therefore \varphi = \sum_n \langle \varphi, \varphi_{1,n} \rangle \varphi_{1,n} = \sum_n \underbrace{\langle \varphi, \varphi_{1,n} \rangle}_{h(n).} 2^{1/2} \varphi(2x-n)$

$$\varphi(x) = \sum_n h(n)\, D_2 T_n \varphi(x)$$

$$\hat{\varphi}(\gamma) = \sum_n h(n)\, \underbrace{D_{2^{-1}} E_{-n} \hat{\varphi}(\gamma)}$$

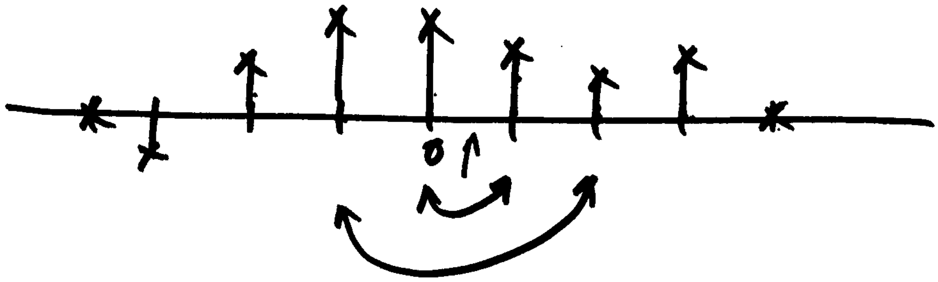
$$D_{2^{-1}}\left(e^{-2\pi i n \gamma} \hat{\varphi}(\gamma)\right)$$

$$= 2^{-1/2} e^{-2\pi i n(\gamma/2)} \hat{\varphi}(\gamma/2)$$

$$= \left(2^{-1/2} \sum_n h(n)\, e^{-2\pi i n(\gamma/2)}\right) \hat{\varphi}(\gamma/2)$$

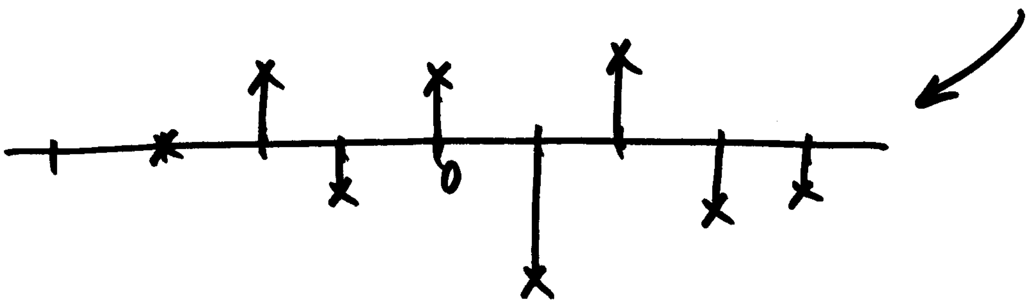
$$= m_0(\gamma/2)\, \hat{\varphi}(\gamma/2).$$

Refer to $\{h(n)\}$ as a <u>scaling filter</u>



$h(k)$

$g(k).$



$$\hat{\psi}(\gamma) = \underbrace{\left(2^{-1/2} \sum_n g(n) e^{-2\pi i n(\gamma/2)}\right)} \hat{\varphi}(\gamma/2).$$

$$m_1(\gamma) = 2^{-1/2} \sum_n g(n)\, e^{-2\pi i n \gamma} = 2^{-1/2} \sum_n (-1)^n\, \overline{h(1-n)}\, e^{-2\pi i n \gamma}$$

$$= \overline{2^{-1/2} \sum_n (-1)^n h(1-n)\, e^{2\pi i n \gamma}}$$

$$= 2^{-1/2}\, \overline{-\sum_m e^{-\pi i m} h(m)\, e^{2\pi i (1-m)\gamma}}$$

$$= \overline{2^{-1/2} e^{2\pi i \gamma} \sum_m h(m)\, e^{-2\pi i m(\gamma + 1/2)}}$$

$$= 2^{-1/2} e^{-2\pi i (\gamma + 1/2)}\, \overline{\sum_m h(m)\, e^{-2\pi i m (\gamma + 1/2)}}$$

$$= e^{-2\pi i (\gamma + 1/2)}\, \overline{m_0(\gamma + 1/2)}$$

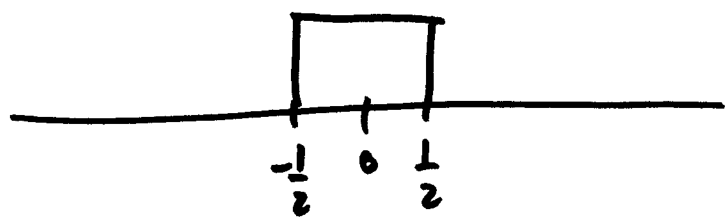
$m = 1-n$

$n = 1-m$

$(-1)^n = -(-1)^m$

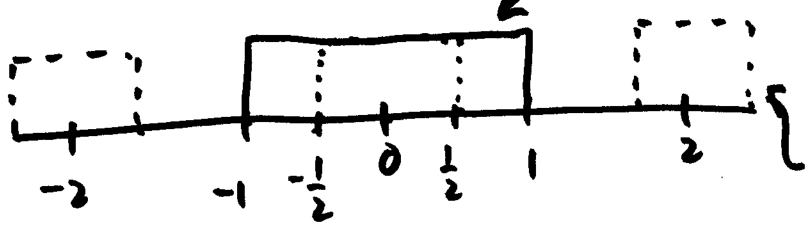
$= -e^{-\pi i m}$

$\hat{\varphi}(\gamma)$



$\hat{\varphi}(\gamma/2)$

$m_0(\gamma/2)$



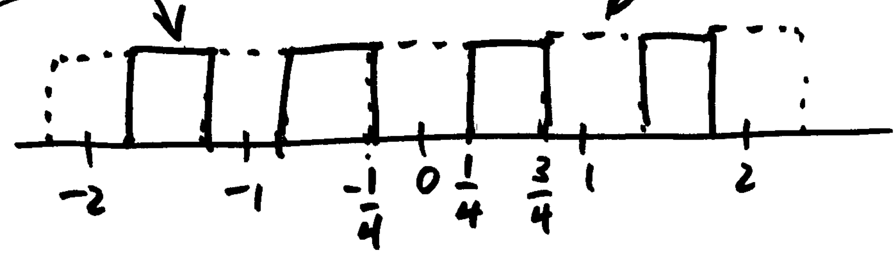
So $\hat{\varphi}(\gamma) = m_0(\gamma/2)\,\hat{\varphi}(\gamma/2)$

where $m_0(\gamma)$ = period 1 extension of $\chi_{[-1/4,\,1/4]}$.

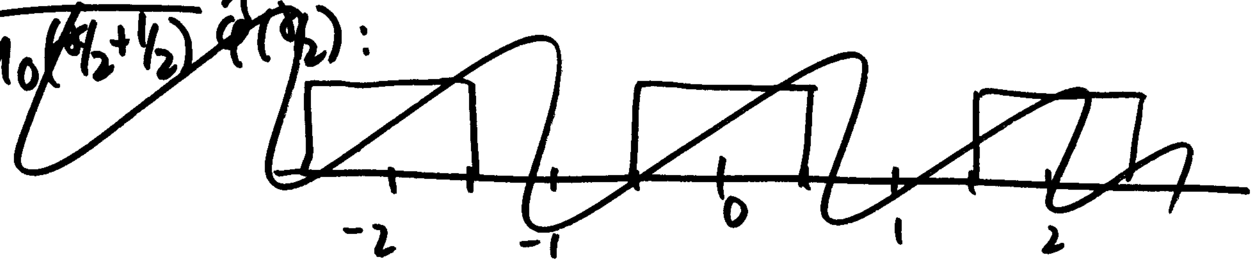
$m_1(\gamma): \; e^{-2\pi i(\gamma+1/2)}\,\overline{m_0(\gamma+1/2)}$

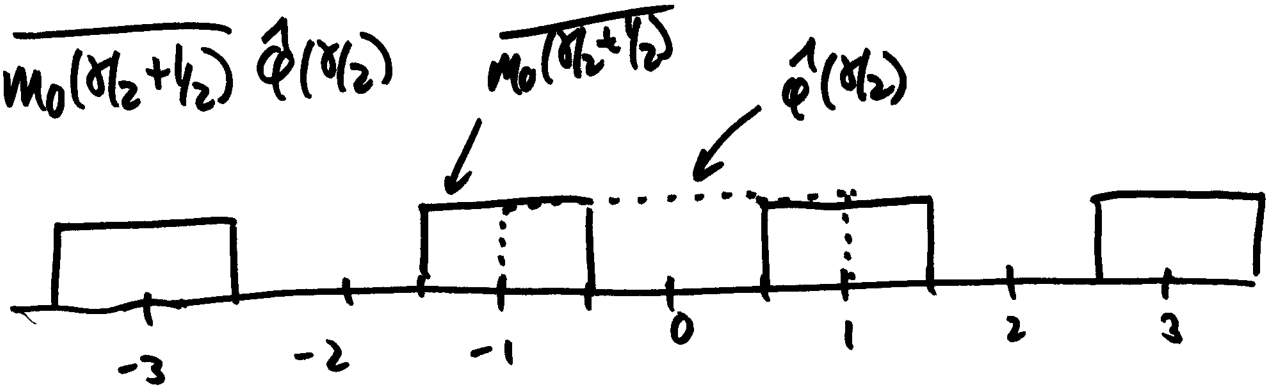
$m_0(\gamma)$

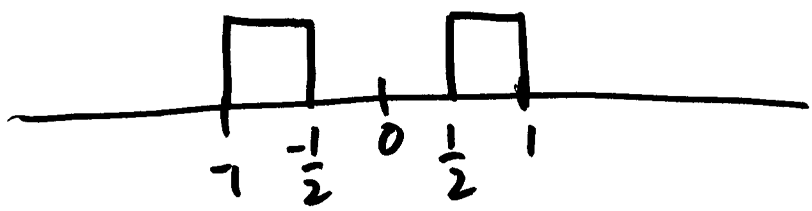
$\overline{m_0(\gamma+1/2)}$:



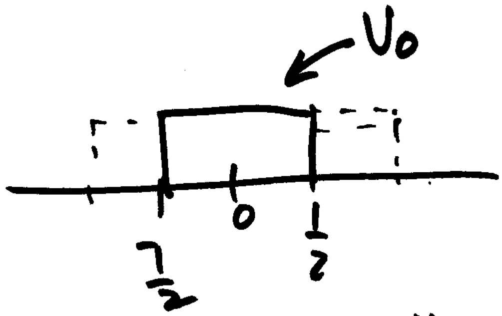
$\overline{m_0(\gamma/2+1/2)}\,\hat{\varphi}(\gamma/2)$:



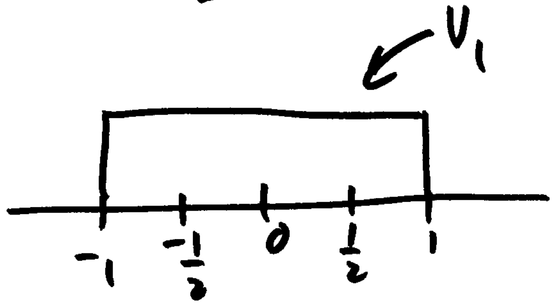


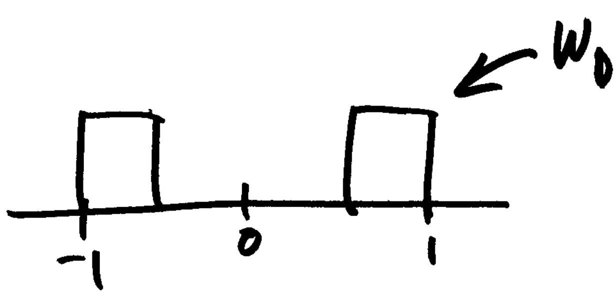


$|\hat{\psi}(\gamma)|$



$|\hat{\varphi}(\gamma)|$

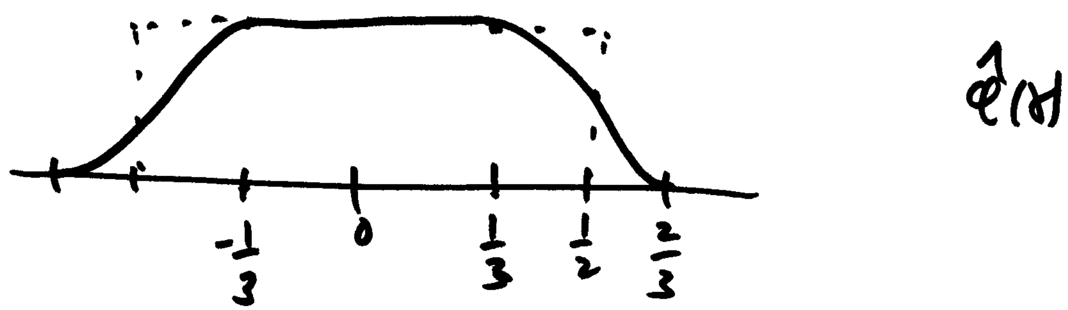


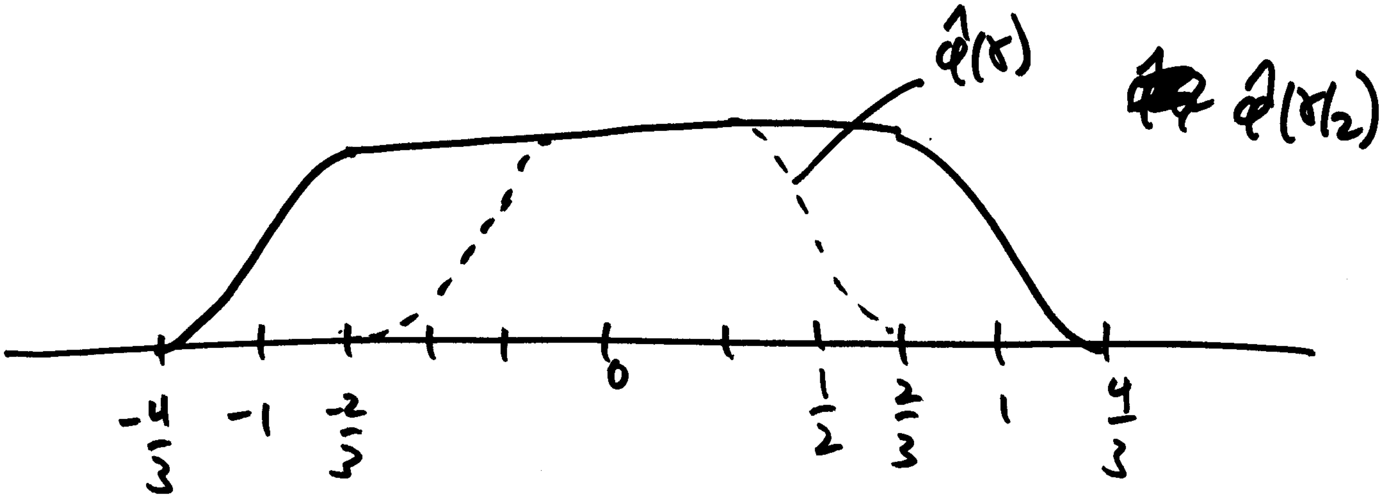


$V_1 = V_0 \oplus W_0$

$\overline{\text{span}}\{T_n\psi\} = W_0$ Problem: Find such a ψ.

(c)



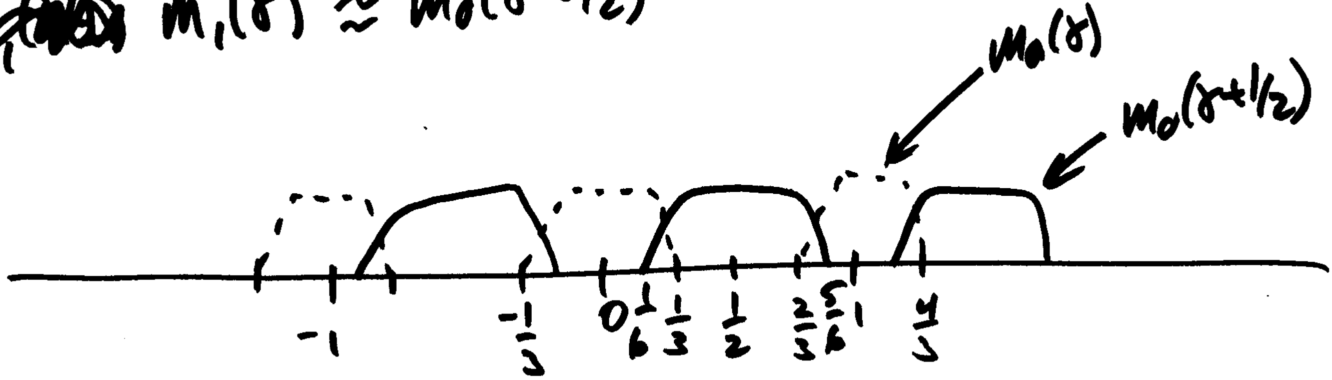


$\hat{\varphi}(\gamma) = m_0(\gamma/2)\,\hat{\varphi}(\gamma/2)$ where

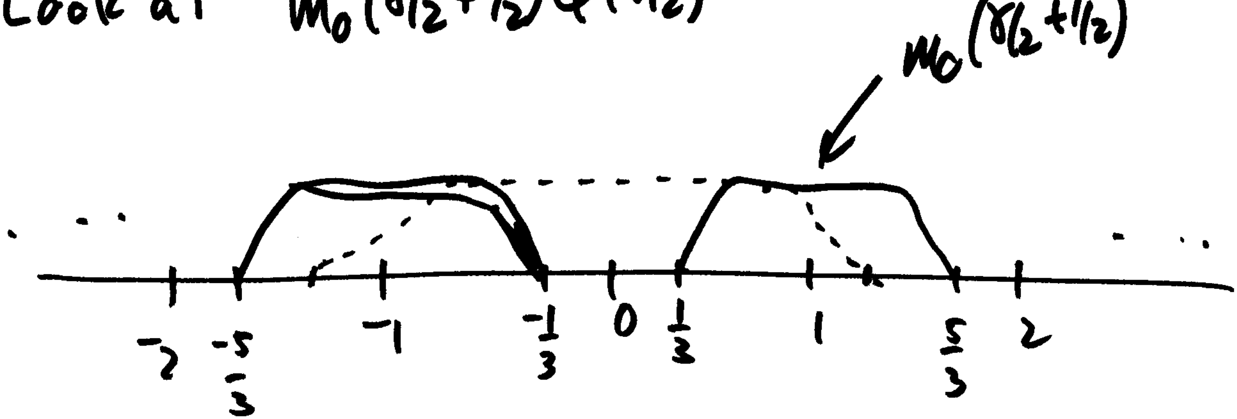
$m_0(\gamma)$ is the period 1 extension

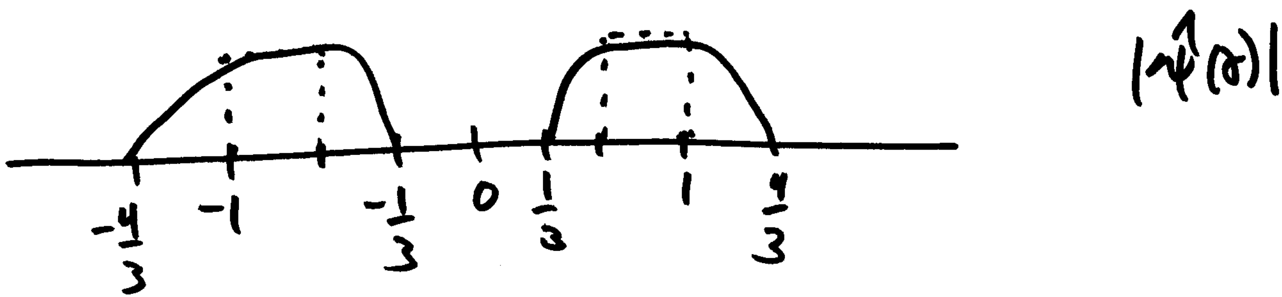
of $\hat{\varphi}(2\gamma)$.

$m_1(\gamma) \approx m_0(\gamma + 1/2)$



Look at $m_0(\gamma/2+1/2)\hat{\varphi}(\gamma/2)$



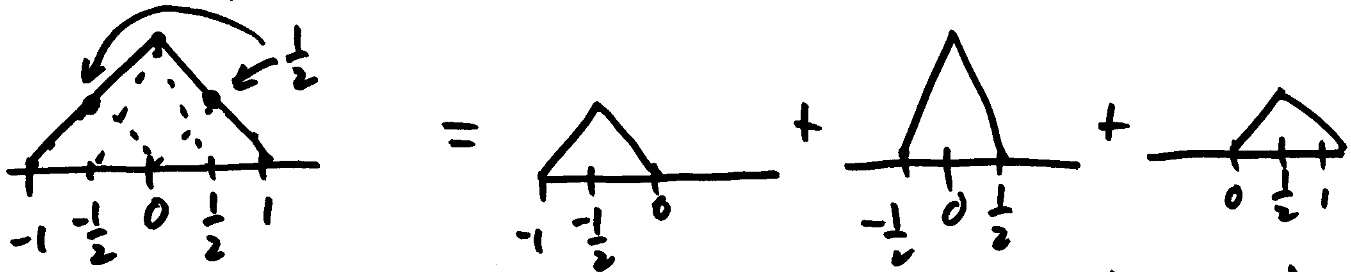


$$\varphi(x) = (1-|x|)\chi_{[-1,1]} \longrightarrow \tilde{\varphi}(x)$$

want $m_0(\gamma)$ so that

$$\hat{\tilde{\varphi}}(\gamma) = m_0(\gamma/2)\,\hat{\tilde{\varphi}}(\gamma/2)$$

Already know:



$$\varphi(x) = \frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1)$$

$$= \frac{1}{2\sqrt{2}}\varphi_{1,-1} + \frac{1}{\sqrt{2}}\varphi_{1,0} + \frac{1}{2\sqrt{2}}\varphi_{1,1}$$

$$\hat{\varphi}(\gamma) = \cos^2(\pi\gamma/2)\,\hat{\varphi}(\gamma/2)$$

Recall:
$$\hat{\tilde{\varphi}}(\gamma) = \Big(\sum_n |\hat{\varphi}(\gamma+n)|^2\Big)^{-1/2}\hat{\varphi}(\gamma)$$

$$= \Big(\frac{1}{3}(1+2\cos^2(\pi\gamma))\Big)^{-1/2}\hat{\varphi}(\gamma)$$

$$\therefore\ \hat{\tilde{\varphi}}(\gamma) = \frac{\sqrt{3}}{(1+2\cos^2\pi\gamma)^{1/2}}\cos^2\Big(\frac{\pi\gamma}{2}\Big)\,\hat{\varphi}(\gamma/2)$$

$m_0(\gamma/2)$

$$= \boxed{\frac{\sqrt{3}\cos^2\frac{\pi\gamma}{2}}{(1+2\cos^2\pi\gamma)^{1/2}}\,\frac{1}{\sqrt{3}}\Big(1+2\cos^2\Big(\frac{\pi\gamma}{2}\Big)\Big)^{1/2}}\;\hat{\tilde{\varphi}}(\gamma/2)$$

$$\hat{\psi}(\gamma) = m_1(\gamma/2)\,\hat{\tilde{\varphi}}(\gamma/2)$$

$$= \underbrace{m_1(\gamma/2)\,\frac{\sqrt{3}}{\left(1+2\cos^2\left(\frac{\pi\gamma}{2}\right)\right)^{1/2}}}_{d(\gamma/2).}\,\hat{\varphi}(\gamma/2).$$

$$\psi(x) = \sum \underrightarrow{d(n)}\, 2^{1/2}\varphi(2x-n)$$