10-9-03

Fourier analysis

Periodic functions

Discrete functions (signals)

functions on $\mathbb{R}$

Periodic discrete ftns (DFT)

Will construct wavelet bases with nice properties

Haar

Good

localized in time

easy to implement

"detect" jump discont.

Bad

discontinuous

do not distinguish frequencies well.

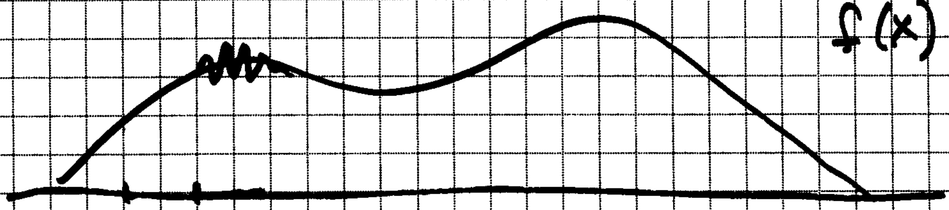
do not "detect" corners.

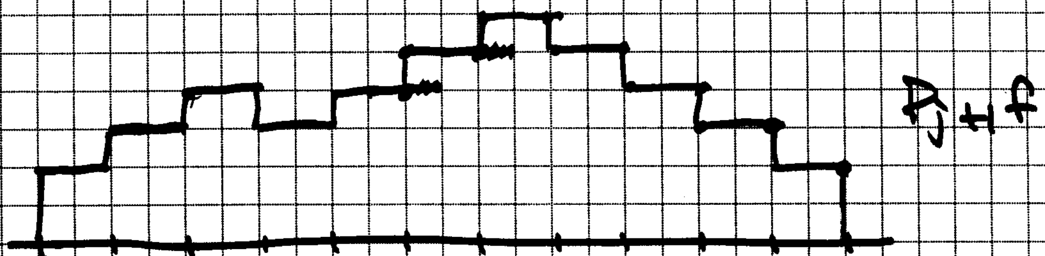
<u>Remark</u>: (a) $P_j f(x)$ is a scale j dyadic step function

(b) $\{p_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system. and $P_j f$ is the Fourier series of f ~~and~~ in $\{p_{jk}\}_{k\in\mathbb{Z}}$.

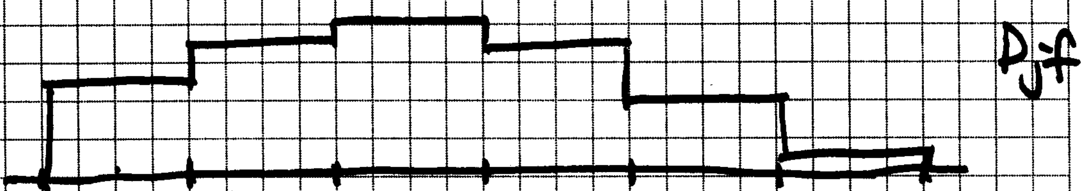
(c) By Best Approx. Lemma, $P_j f$ is the scale j dyad. step ftn closest to f in L^2 sense. That is, $P_j f$ minimizes $\|f - P_j f\|_2$ over all scale j dyad step ftns.

(d) $V_j = \overline{\text{span}}\{p_{jk}\}_{k\in\mathbb{Z}}$ consists of all scale j dyadic step ftns in $L^2(\mathbb{R})$.





$P_{j+1}f$ contains no details in f smaller than $2^{-(j+1)}$ in size



$P_j f$ contains no details smaller than 2^{-j}.

$Q_j f = P_{j+1}f - P_j f$ contains the details in f visible at scale $2^{-(j+1)}$ but not visible at scale 2^{-j}

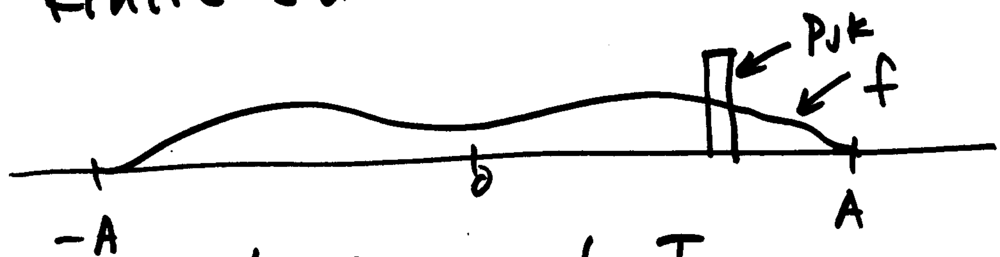
Proof of first Theorem: (part (b) only)

Need to show (1) $\{h_{jk}\}_{j,k\in\mathbb{Z}}$ is orthonormal and (2) $\{h_{jk}\}_{j,k\in\mathbb{Z}}$ is complete. (1) is already done. Only (2) remains.

We will show that if $f \in C_c^0(\mathbb{R})$ then

$$f = \sum_{j,k} \langle f, h_{jk}\rangle h_{jk}.$$

Let $f \in C_c^0(\mathbb{R})$. Note that $P_j f = \sum_k \langle f, p_{jk}\rangle p_{jk}$ is a finite sum for all j.



We can write for each J,

$$\begin{aligned} P_J f &= (P_J f - P_{J-1} f) + (P_{J-1} f - P_{J-2} f) + \cdots + (P_{-J+1} f - P_{-J} f) + P_{-J} f \\ &= Q_{J-1} f + Q_{J-2} f + \cdots + Q_{-J} f + P_{-J} f \\ &= \sum_{j=-J}^{J-1} Q_j f + P_{-J} f \\ &= \sum_{j=-J}^{J-1} \sum_k \langle f, h_{jk}\rangle h_{jk} + P_{-J} f. \end{aligned}$$

by second Thm part (c). Note this sum is also finite.

Next note that

$$\sum_{j=-J}^{J-1} \sum_{k} \langle f, h_{jk} \rangle h_{jk} = P_J f - P_{-J} f$$

Remains to show that $P_J f - P_{-J} f \to f$ as $J \to \infty$, in the L^2 norm. But.

$$\| f - P_J f + P_{-J} f \|_2 \leq \| f - P_J f \|_2 + \| P_{-J} f \|_2$$

By parts (a) and (b) of second Thm, Right side $\to 0$ as $J \to \infty$.

$$\therefore \quad f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, h_{jk} \rangle h_{jk}. \qquad \blacksquare$$

The Haar MRA.

$$\varphi(x) = p_{00}(x) = \mathbb{1}_{[0,1)}(x)$$

$\{T_n \varphi\}_{n\in\mathbb{Z}} = \{\mathbb{1}_{[n,n+1)}\}_{n\in\mathbb{Z}}$ is an o.n. system.

$$V_0 = \overline{\text{span}}\{\mathbb{1}_{[n,n+1)}\}_{n\in\mathbb{Z}}$$

$$= \{\text{scale } 0 \text{ step ftns in } L^2\}.$$

$$V_j = \{f : D_{2^{-j}} f \in V_0\}.$$

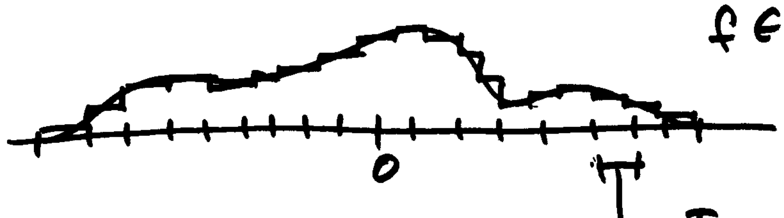
$$= \{\text{scale } j \text{ dyadic step ftns in } L^2\}.$$

Verify properties of MRA:

(a) $V_j \subseteq V_{j+1}$ (Yes. Homework exercise)

(b) $L^2 = \overline{\text{span}}\{V_j\}_{j\in\mathbb{Z}}$

$f \in C_c^0(\mathbb{R})$

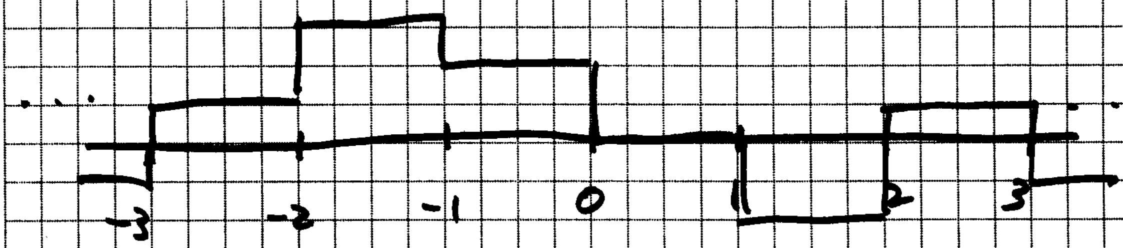


I_{jk} <u>j large</u>

Intuitively clear that $P_j f \to f$ as $j \to \infty$.

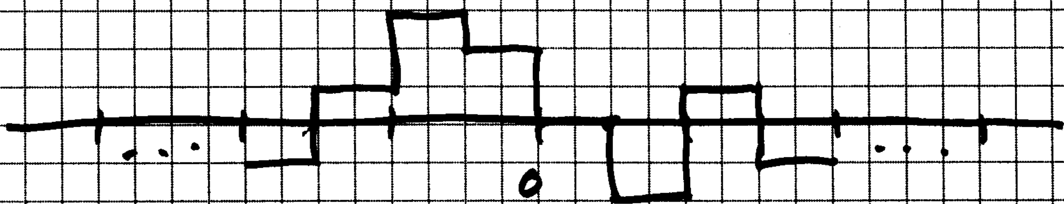
(c) $\bigcap V_j = \{0\}$

If $f \in V_j$ for <u>every</u> V_j at once, then f must be constant on $[0,\infty)$ and $(-\infty,0)$. Since $f \in L^2(\mathbb{R})$ these constants must be <u>zero</u>.

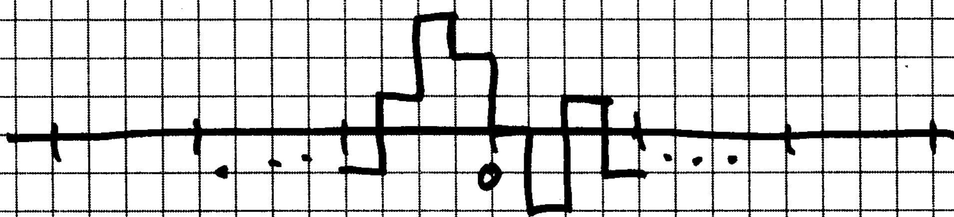


$f \in V_0$

$f(2x) \in V_1$



$f(4x) \in V_2$



(d) True by construction.

(e) Also true by construction.

Band limited MRA.

Define $V_0 = \{ f \in L^2(\mathbb{R}) : \hat{f}(\gamma) = 0 \text{ if } |\gamma| > \frac{1}{2} \}$.

$V_j = ?$ Look at V_1.

$f \in V_0 \iff D_2 f \in V_1$ or $f \in V_1 \iff D_{\frac{1}{2}} f \in V_0$

$\cancel{f \in V_0 \iff \hat{f}(\gamma) = 0 \text{ for } |\gamma| > \frac{1}{2}}$

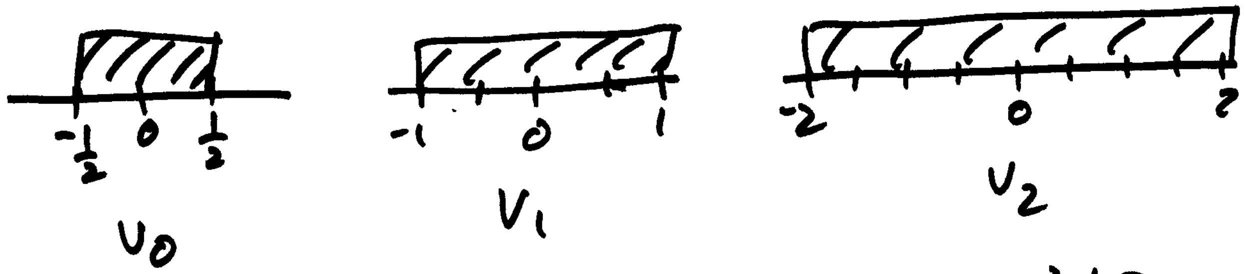
$D_{\frac{1}{2}} f \in V_0 \iff \widehat{D_{\frac{1}{2}} f}(\gamma) = 0 \text{ if } |\gamma| > \frac{1}{2}$

$\iff D_2 \hat{f}(\gamma) = 2^{1/2} \hat{f}(2\gamma) = 0 \text{ if } |\gamma| > \frac{1}{2}$

$\hat{f}(2\gamma) = 0$ if $|\gamma| > \frac{1}{2}$ means

$\hat{f}(u) = 0$ if $|\frac{u}{2}| > \frac{1}{2}$ or $|u| > 1$

$\therefore V_1 = \{ f \in L^2(\mathbb{R}) : \hat{f}(\gamma) = 0 \text{ if } |\gamma| > 1 \}$.



$\therefore V_j = \{ f \in L^2(\mathbb{R}) : \hat{f}(\gamma) = 0 \text{ if } |\gamma| > 2^{j-1} \}$.

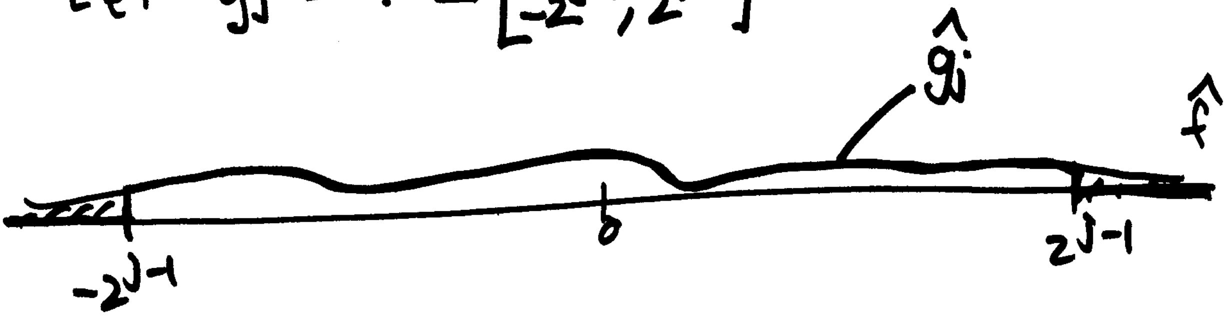
verify ~~def~~ properties of MRA.

(a) $V_j \subseteq V_{j+1}$ clear

(b) $L^2 = \overline{\text{span}\{V_j\}}$.

If $f \in L^2(\mathbb{R})$ then $\hat{f} \in L^2(\mathbb{R})$

Let $\hat{g}_j = \hat{f}\, \mathbb{1}_{[-2^{j-1}, 2^{j-1}]}$



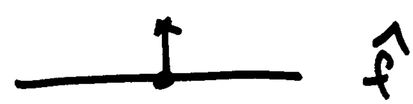
$\|\hat{f} - \hat{g}_j\|_2 \to 0$ as $j \to \infty$.

But $g_j \in V_j$ and by Plancherel $\|f - g_j\|_2 \to 0$ as $j \to \infty$.

(c) $\bigcap V_j = \{0\}$.

If $f \in V_j$ for all j at once then

$\hat{f}(\gamma) = 0$ for all $|\gamma| > 0$



Since $\hat{f} \in L^2$, $\hat{f} \equiv 0$ so $f \equiv 0$.

(d) Satisfied by definition

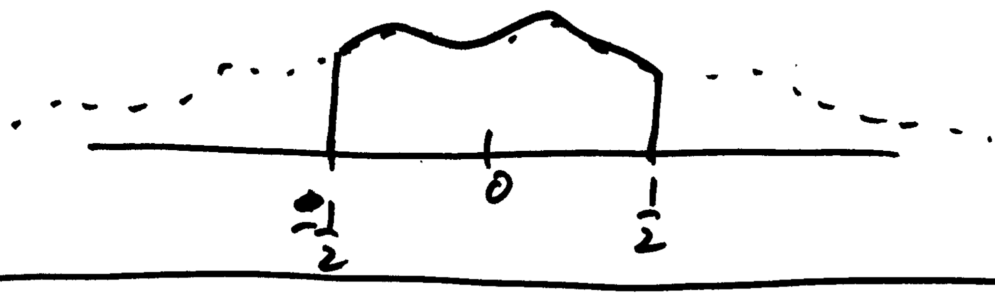
(e) Take $\varphi(x) = \dfrac{\sin \pi x}{\pi x}$. Shannons formula says.

If $f \in V_0$ then $f(x) = \sum_n f(n) \dfrac{\sin \pi(x-n)}{\pi(x-n)} = \underline{\sum_n f(n)\varphi(x-n)}$

BL MRA:

$$f \in V_0 \longrightarrow f = \sum f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}$$

$$\hat{f}(\gamma) = \sum f(n) e^{-2\pi i n \gamma} 1_{[-1/2,1/2]} \leftarrow$$



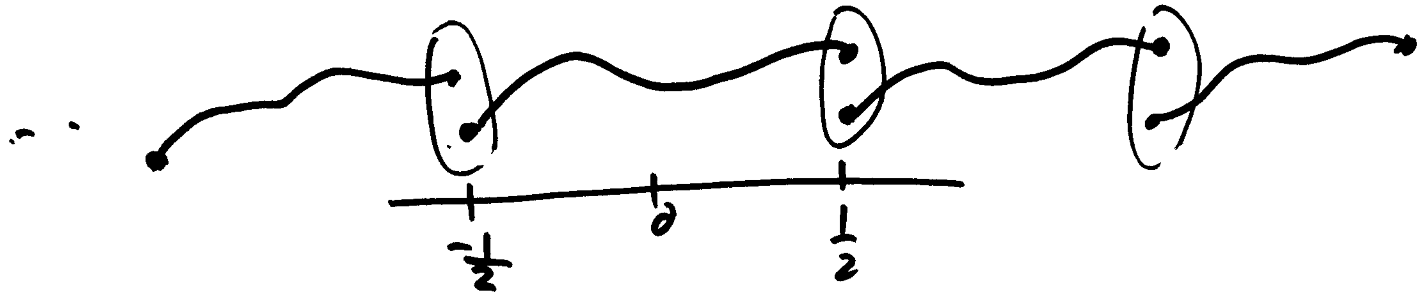
Taking an arbitrary f in L^2.

Want to project f onto V_0 (This will be $P_0 f$).

We cut-off $\hat{f} \longrightarrow \hat{f}(\gamma) 1_{[-1/2,1/2)}(\gamma)$

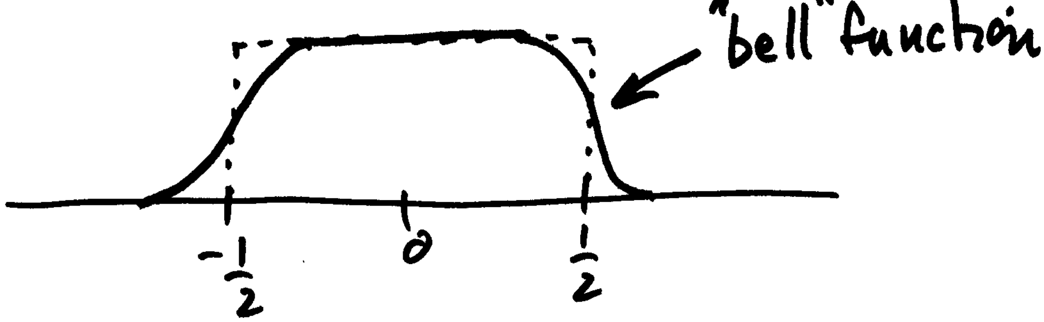
Then take Fourier series $\longrightarrow \widehat{P_0 f}(\gamma) = \sum_n c(n) e^{-2\pi i n \gamma} 1_{[-1/2,1/2)}$

problem: If $\hat{f}(-1/2) \neq \hat{f}(\tfrac{1}{2})$ then the $c(n)$ will be too large.

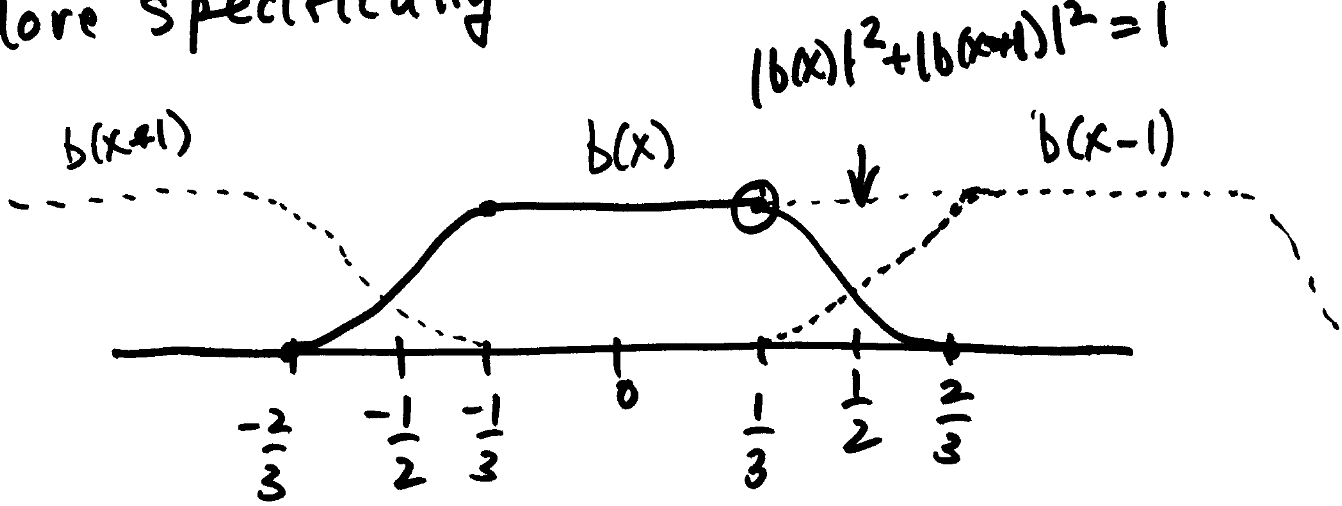


Meyer's idea:

Replace $1_{[-1/2,1/2]}(\gamma)$ by a smooth cut-off function in frequency



More specifically



$\varphi(x)$ satisfies $\hat{\varphi}(\gamma)=b(\gamma)$ some C^k bell b.

Define $V_0=\overline{\text{span}}\{T_n\varphi\}$.

Define V_j by $f\in V_j \Leftrightarrow D_{2^{-j}}f\in V_0$.

Go from there to verify (a)–(e) in defn of MRA.

Results about collections $\{T_n g\}_{n\in\mathbb{Z}}$.

(a) First part follows from known results.

Second part: $f \in \overline{\operatorname{span}}\{T_n g\} \iff$

$$f(x) = \sum_n c(n)\, g(x-n) \iff$$

$$\hat{f}(\gamma) = \sum_n c(n)\, e^{-2\pi i n\gamma}\,\hat{g}(\gamma)$$

$$= \hat{g}(\gamma)\left(\sum_n c(n)\, e^{-2\pi i n\gamma}\right)$$

$$= \hat{g}(\gamma)\,\hat{c}(\gamma)$$

(b) Spse $\{T_n g\}$ is an o.n. system. Look at

$$\langle T_m g, T_n g\rangle = \langle e^{-2\pi i m\gamma}\hat{g}(\gamma), e^{-2\pi i n\gamma}\hat{g}(\gamma)\rangle$$

$$= \int_{-\infty}^{\infty} \hat{g}(\gamma)\, e^{-2\pi i m\gamma}\, \overline{\hat{g}(\gamma)}\, e^{2\pi i n\gamma}\, d\gamma$$

$$= \int_{-\infty}^{\infty} |\hat{g}(\gamma)|^2\, e^{-2\pi i (m-n)\gamma}\, d\gamma$$

$$= \sum_k \int_k^{k+1} |\hat{g}(\gamma)|^2 e^{-2\pi i (m-n)\gamma}\, d\gamma \qquad \begin{array}{l} u = \gamma - k \\ du = d\gamma \\ \gamma = u + k \end{array}$$

$$= \sum_k \int_0^1 |\hat{g}(u+k)|^2 e^{-2\pi i (m-n) u}\, du$$

$$= \int_0^1 \left(\sum_k |\hat{g}(u+k)|^2\right) e^{-2\pi i (m-n) u}\, du$$

This is the $(m-n)^{th}$ Fourier coeff of the periodic function $\sum_k |\hat{g}(u+k)|^2$.

So $\langle T_m g, T_n g\rangle = \delta_{nm} \iff \langle g, T_\ell g\rangle = \delta_{0\ell}$

$\iff$ Fourier coeffs of $\sum |\hat{g}(u+k)|^2$ equal $\delta_{0\ell}$.

$\iff \sum_k |\hat{g}(u+k)|^2 \equiv 1$. (because 1 is the only period 1 ftn whose Fourier coeffs are $\begin{cases} 1 & \ell=0 \\ 0 & \ell\neq 0 \end{cases}$.

Verify the Meyer MRA.

First of all: by construction $\{T_n\varphi\}$ is an o.n. system. So (e) is satisfied.

By definition ($f\in V_j \iff D_{2^{-j}}f\in V_0$), (d) is satisfied

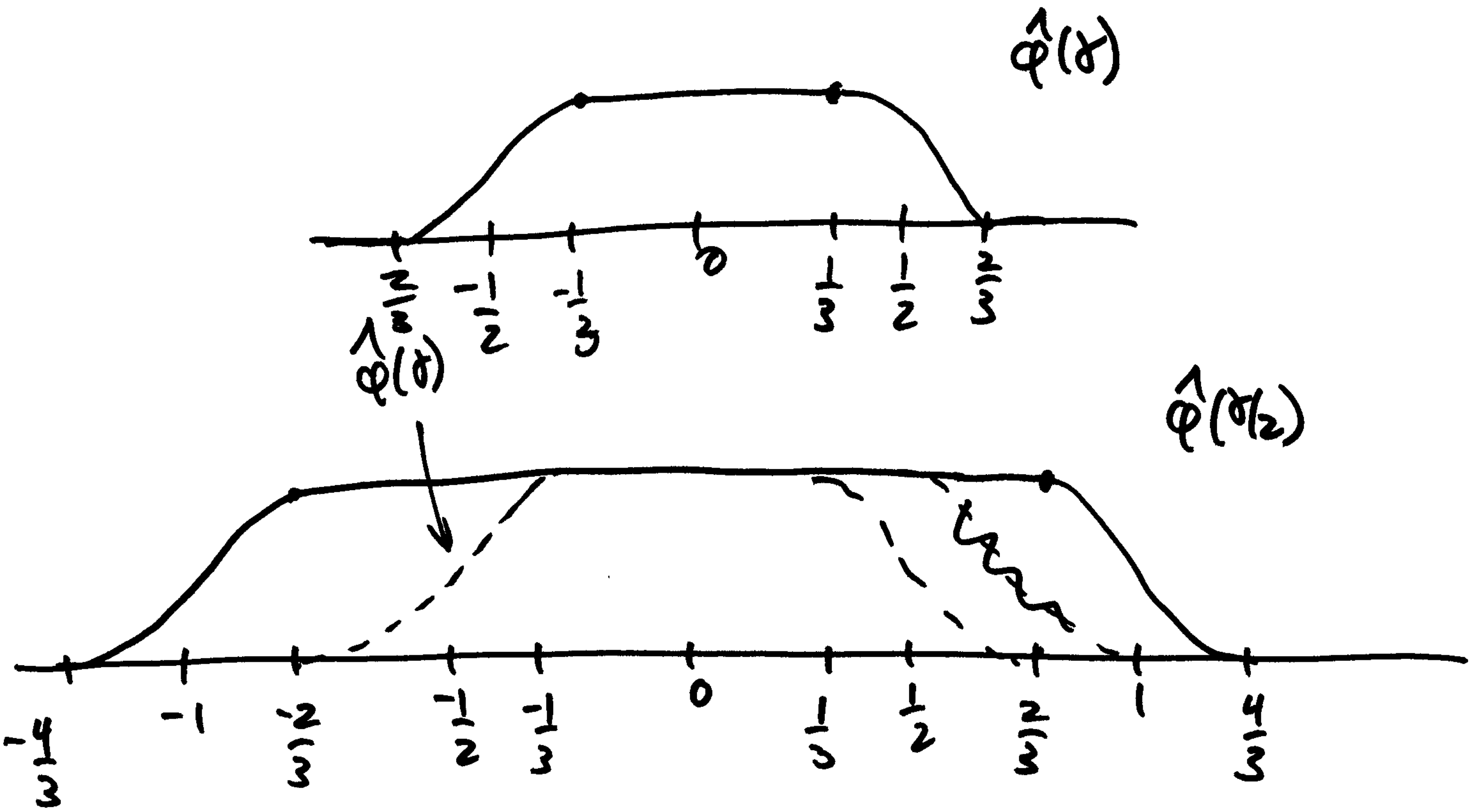
What about (a)? Is $V_0 \subseteq V_1$?

$f\in V_0 \iff f\in \overline{\text{span}}\{T_n\varphi\} \iff \hat{f}(\gamma) = \hat{c}(\gamma)\hat{\varphi}(\gamma)$ ← period 1, ← nice ftn.

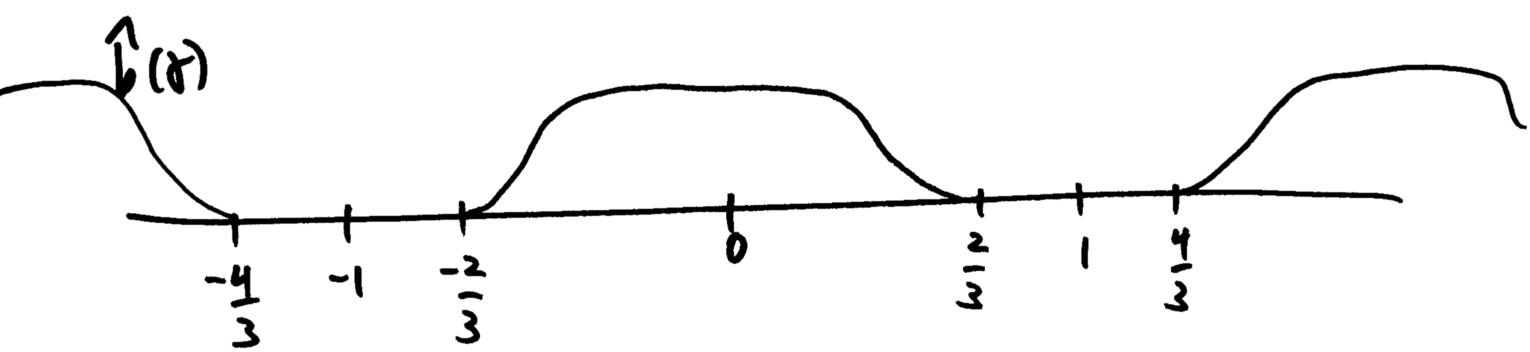
$f\in V_1 \iff D_{2^{-1}}f\in V_0 \iff D_2\hat{f}(\gamma) = \hat{a}(\gamma)\hat{\varphi}(\gamma)$ ← period 1

$\iff \hat{f}(2\gamma) = \hat{a}(\gamma)\hat{\varphi}(\gamma)$

$\iff \hat{f}(\gamma) = \hat{a}(\gamma/2)\hat{\varphi}(\gamma/2) = \hat{b}(\gamma)\hat{\varphi}(\gamma/2)$ ← period 2



Let $\hat b(\gamma)$ = period 2 extension of $\hat\varphi(\gamma)$.



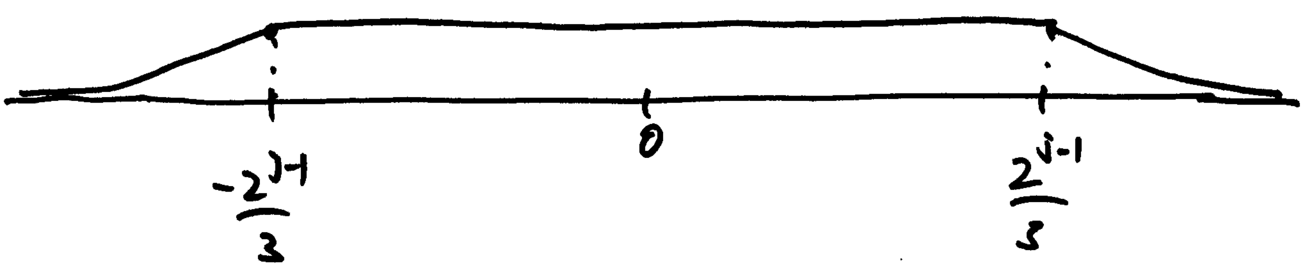
Clearly $\hat\varphi(\gamma) = \hat b(\gamma)\,\hat\varphi(\gamma/2)$.

$\therefore\ f \in V_0 \Rightarrow \hat f = \hat c(\gamma)\,\hat\varphi(\gamma) \Rightarrow \hat f = \underbrace{\underset{\text{period }1}{\hat c(\gamma)}\ \underset{\text{period }2}{\hat b(\gamma)}}_{\text{period }2}\,\hat\varphi(\gamma/2)$

$\therefore\ f \in V_1$ and $V_0 \subseteq V_1$!

Finally need to verify (b) and (c).

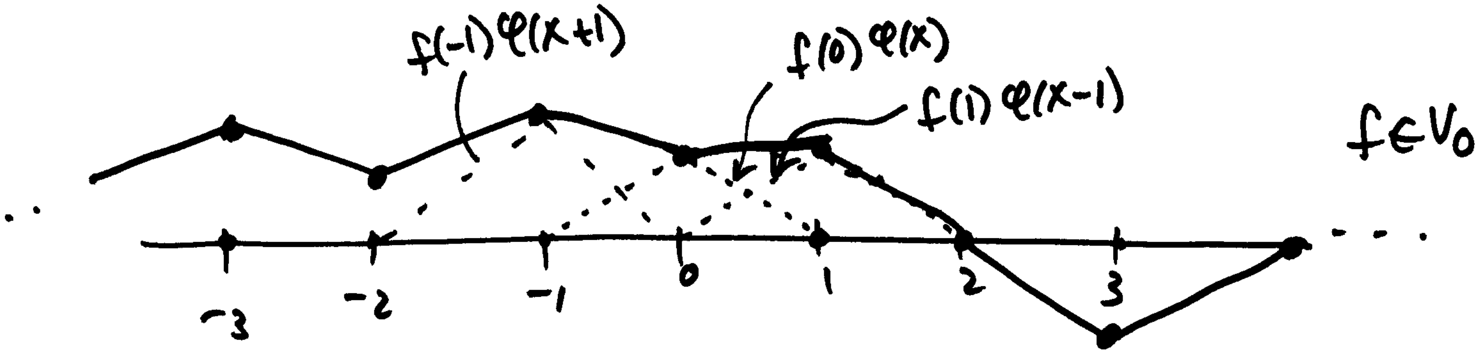
For (b):



By choosing j large enough I can find a function $g \in V_j$ such that
$\hat{g}(\gamma) = \hat{f}(\gamma)$ on the interval $\left[-\frac{1}{3}2^{j-1}, \frac{1}{3}2^{j-1}\right]$.

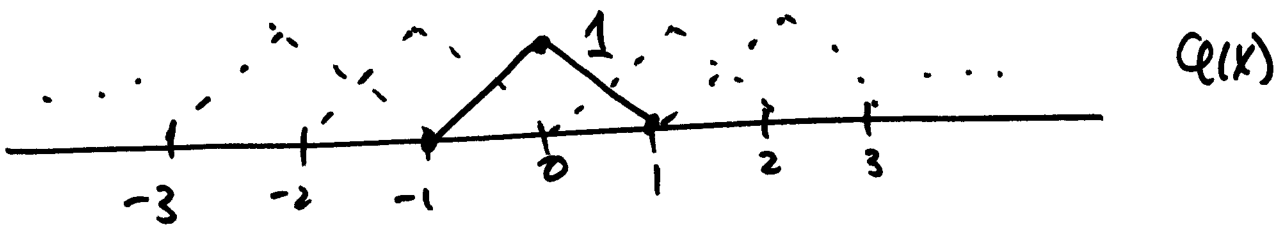
So as $j \to \infty$ I can get as close as desired to $\hat{f}$.

(c) is as in the BL MRA case.



$I_{0,k} = [k, k+1)$.

Not clear that there is a scaling function.



<u>Clear</u>: any $f \in V_0$ can be written

$$f(x) = \sum_n f(n)\,\varphi(x-n) \quad \underline{\text{pointwise}}$$

Problem: $\{\varphi(x-n)\}$ is not orthonormal!

This can be fixed.