10-2-03

Fourier Analysis

Fourier series $f = \frac{1}{a}\sum \langle f, e_{n/a}\rangle e_{n/a}$

General orthonormal systems

coefficients; convergence; completeness/uniqueness

Bessel's inequality

Best Approximation Lemma

Completeness

Equivalent conditions for completeness

span $\{g_n\}$

$\overline{\text{span}}\{g_n\}$

Fourier transforms

Fourier inversion

convolution

Smoothness vs. Decay.

Sampling Theorem.

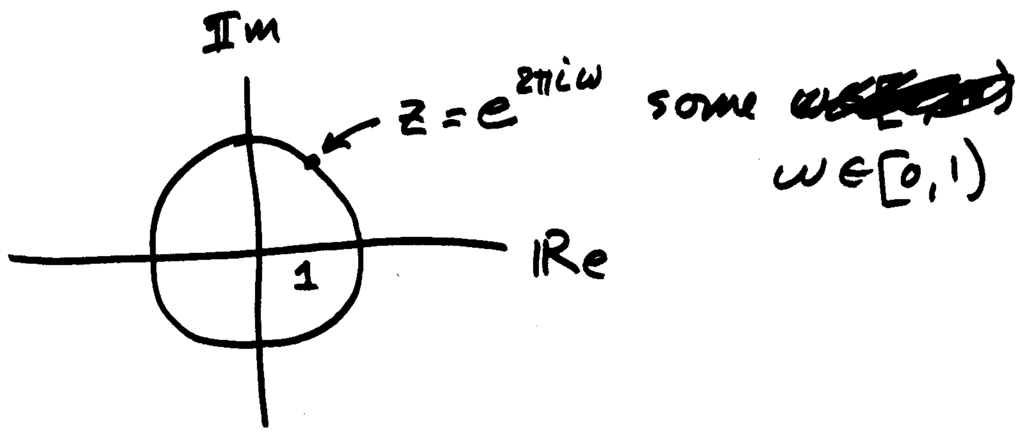
e.g. $x(n) = \frac{1}{n^2}$; $x(0) = 0$

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2} < \infty$$

e.g. $x(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$ some $|a| < 1$

$$\sum_{n \in \mathbb{Z}} x(n) = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

e.g. $x(n) = \delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$



$$\hat{x}(\omega) = \sum x(n) e^{-2\pi i n \omega} = X(e^{2\pi i \omega}) \equiv X(z)$$

where $|z| = 1$ in $\mathbb{C}$

$$X(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n} \leftarrow \text{power series in } z^{-1}$$

Power series have a radius of convergence $R > 0$. In our case

$$\sum_{n=0}^{\infty} x(n) z^{-n}$$ will converge if $|z^{-1}| < R$

or $|z| > \frac{1}{R}$

and $$\sum_{n=-\infty}^{-1} x(n) z^{-n} = \sum_{n=1}^{\infty} x(-n) z^{n}$$

will converge for $|z| < R'$

so $X(z)$ will converge on some annulus $\frac{1}{R} < |z| < R'$

We hope that $\frac{1}{R} < 1 < R'$

Then $X(e^{2\pi i \omega})$ makes sense.

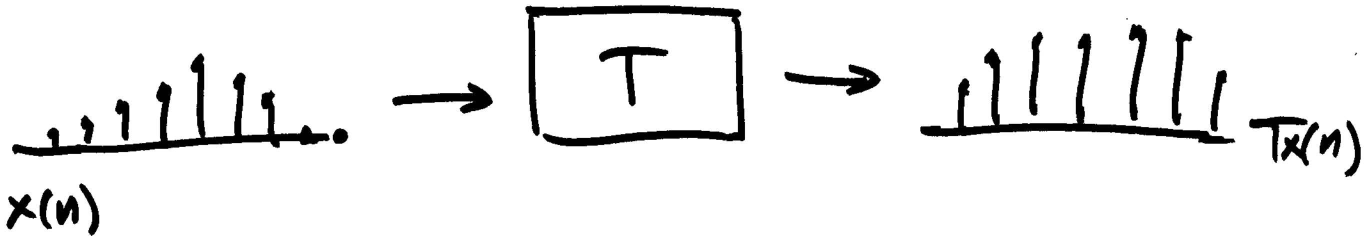
$$x(n) \longrightarrow \boxed{\;T\;} \longrightarrow Tx(n)$$

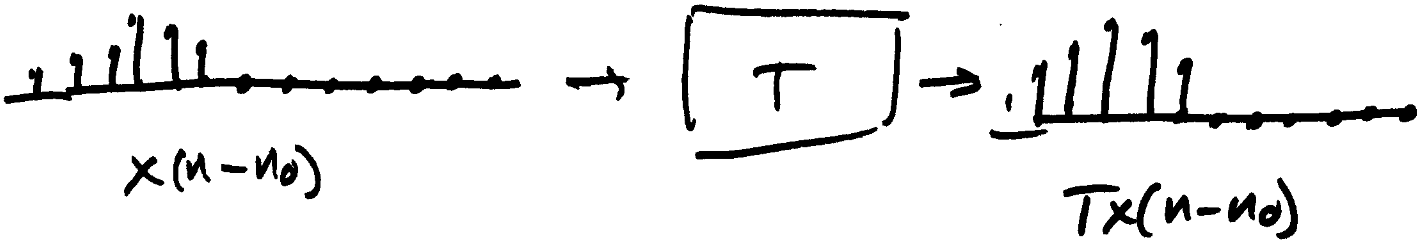
linear
$$\left[\begin{array}{l} x_1(n) + x_2(n) \longrightarrow \boxed{\;T\;} \longrightarrow Tx_1(n) + Tx_2(n) \\[2em] a\,x_1(n) \longrightarrow \boxed{\;T\;} \longrightarrow a\,Tx_1(n) \end{array}\right.$$

$$x_1(n) \longrightarrow \oplus \longrightarrow \tilde{x}(n) \longrightarrow \boxed{\;T\;} \longrightarrow T\tilde{x}(n)$$
$$\uparrow$$
noise

$$\text{noise } \text{~~power~~ intensity} = \sum |\tilde{x}(n) - x(n)|$$

$$\begin{aligned} \text{error} &= \sum |T\tilde{x}(n) - Tx(n)| \\ &= \sum |T(\tilde{x} - x)(n)| \\ &\leq c \sum |\tilde{x}(n) - x(n)| = \text{noise intensity} \end{aligned}$$


$x(n)$
T
$Tx(n)$


$x(n-n_0)$
T
$Tx(n-n_0)$

pf: (a) $$\sum_n |y(n)| = \sum_n |x_1 * x_2(n)|$$

$$= \sum_n \Big| \sum_k x_1(k)\, x_2(n-k) \Big|$$

$$\le \sum_n \sum_k |x_1(k)|\, |x_2(n-k)|$$

$$= \sum_k |x_1(k)| \underbrace{\sum_n |x_2(n-k)|}_{= \sum_n |x_2(n)|}$$

$$= \sum_k |x_1(k)| \sum_n |x_2(n)| < \infty$$

(c) $T_h x(n) = x * h(n)$

stable: $$\sum_n |Tx(n)| \le \underbrace{\left(\sum_n |h(n)|\right)}_{C} \sum_n |x(n)|$$

linear: obvious

Translation invariant: $T_h(\tau_{n_0} x)(n)$

$$= \sum_k \tau_{n_0} x(k)\, h(n-k) = \sum_k x(k-n_0)\, h(n-k)$$

$m = k - n_0$

$k = m + n_0$

$$= \sum_m x(m)\, h(n-m-n_0)$$

$$= \sum_m x(m)\, h((n-n_0)-m) = T_h x(n-n_0) = \tau_{n_0} T_h x(n)$$

<u>Pf</u>: Let $h(n) = T\delta(n)$, $\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

Note that for any $x(n)$

$$x(n) = (x * \delta)(n) = \sum_k x(k)\,\delta(n-k)$$

$$\therefore\ Tx(n) = T\Big(\sum_k x(k)\,\delta(n-k)\Big)$$

$$= \sum_k x(k)\, T(\tau_k \delta)(n)$$

$$= \sum_k x(k)\, \tau_k (T\delta)(n)$$

$$= \sum_k x(k)\, \tau_k h(n)$$

$$= \sum_k x(k)\, h(n-k)$$

Why frequency response?

Let $x(n) = e^{2\pi i n\omega}$, i.e. x is a <u>pure frequency</u>.

$$
\begin{aligned}
Tx(n) &= \sum_k x(k)\,h(n-k) \\
&= \sum_k e^{2\pi i k\omega}\, h(n-k) \\
&= \sum_k e^{2\pi i (n-k)\omega}\, h(k) \\
&= e^{2\pi i n\omega} \sum_k e^{-2\pi i k\omega}\, h(k) \\
&= x(n)\, \underline{\hat{h}(\omega)}
\end{aligned}
$$

Think of $x(n) = \int_0^1 \hat{x}(\omega)\, e^{2\pi i n\omega}\, d\omega$

$$
\begin{aligned}
Tx(n) &= \int_0^1 \hat{x}(\omega)\, T\left(e^{2\pi i n\omega}\right) d\omega \\
&= \int_0^1 \hat{x}(\omega)\, \hat{h}(\omega)\, e^{2\pi i n\omega}\, d\omega
\end{aligned}
$$

$$\therefore\ \widehat{Tx}(\omega) = \hat{x}(\omega)\, \hat{h}(\omega)$$

Pf: $(x * h)(n) = \sum_k x(k) h(n-k)$

$$|(x*h)(n)| = \left|\sum_k x(k) h(n-k)\right|$$

$$\leq \sum_k |x(k)|\,|h(n-k)|$$

$$\leq \max_{0 \leq k \leq N-1} |x(k)| \sum_k |h(n-k)|$$

$$= \max |x(k)| \sum_k |h(k)| < \infty$$

$$x*h(n+N) = \sum_k x(k) h(n+N-k)$$

$$= \sum_k x(k) h(n-(k-N))$$

$m = k-N$

$k = m+N$

$$= \sum_m x(m+N) h(n-m)$$

$$= \sum_m x(m) h(n-m) = x*h(n)$$

$$\hat{x}(\omega) = \sum_{k} x(k)\, e^{-2\pi i n \omega} \leftarrow \frac{n}{N}$$

$$\sum_{k=0}^{N-1}$$

$$\hat{x}(n+N) = \sum_{j=0}^{N-1} x(j)\, e^{-2\pi i j (n+N)/N}$$

$$= \sum_{j=0}^{N-1} x(j)\, e^{-2\pi i j n/N}\, \cancelto{1}{e^{-2\pi i j}}$$

$$= \hat{x}(n)$$

<u>Pf</u>: $\displaystyle \frac{1}{N} \sum_{n=0}^{N-1} \hat{x}(n)\, e^{2\pi i n j/N}$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} x(k)\, e^{-2\pi i n k/N}\, e^{2\pi i n j/N}$$

$$= \sum_{k=0}^{N-1} x(k)\, \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n (j-k)/N}} \leftarrow = 1 \text{ if } j = k$$

$$\frac{1}{N} \sum_{n=0}^{N-1} \left(e^{2\pi i (j-k)/N}\right)^{n} = \frac{1}{N}\, \frac{1 - e^{2\pi i (j-k)}}{1 - e^{2\pi i (j-k)/N}} = 0 \text{ if } j \neq k$$

$$= x(j)$$

Pf:
$$\sum_{j=0}^{N-1} x*h(j)\, e^{-2\pi i n j/N}$$
$$= \sum_{j=0}^{N-1} \sum_{k} x(k)\, h(j-k)\, e^{-2\pi i n j/N}$$
$$= \sum_{j=0}^{N-1} \sum_{k} h(k)\, x(j-k)\, e^{-2\pi i n j/N}$$
$$= \cancel{\sum_{k} h(k)} \quad \cancel{\sum_{j=0}^{N-1} x(n-k)\, e^{-2\pi i (n-k) j/N}}$$
$$= \sum_{k} h(k)\, e^{-2\pi i n k/N} \sum_{j=0}^{N-1} x(j-k)\, e^{-2\pi i n (j-k)/N}$$
$$= \hat{h}\left(\frac{n}{N}\right) \hat{x}(n)$$

Note: $\hat{h}\left(\frac{n}{N}\right)$ is a period N sequence and is the DFT of something. That something is $\tilde{h}(n) = \sum_{k} h(n+kN)$,

the N-periodization of h.
Another calculation shows that
$$(x*h)(n) = \sum_{k=0}^{N-1} x(k)\, \tilde{h}(n-k)$$
which inspires the definition of circular convolution.

<u>Rem!</u> Period N signals can be identified with vectors in $\mathbb{R}^N$.

$$x(n) \longleftrightarrow \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Circ. convolution $\circledast$ can be thought of as a linear transformation from $\mathbb{R}^N$ to $\mathbb{R}^N$.

Pf: (a) $\overline{X(2-k)} = \overline{\sum_{n=1}^{N} x(n)\, e^{-2\pi i(2-k-1)(n-1)/N}}$

$$= \sum_{n=1}^{N} x(n)\, e^{2\pi i(1-k)(n-1)/N}$$

$$= \sum_{n=1}^{N} x(n)\, e^{-2\pi i(k-1)(n-1)/N} = X(k)$$

Note: $X(1)$ is always real.

(b) By (a) I know $X(k) = \overline{X(2-k)}$.

So sufficient to show $X(k)$ is real.

$$\overline{X(k)} = \sum_{n=1}^{N} x(n)\, e^{-2\pi i(k-1)(n-1)/N}$$

$$= \sum_{n=1}^{N} x(n)\, e^{2\pi i(k-1)(n-1)/N}$$

$m = 2-n$

$n = 2-m$

$n=1 \to m=1$

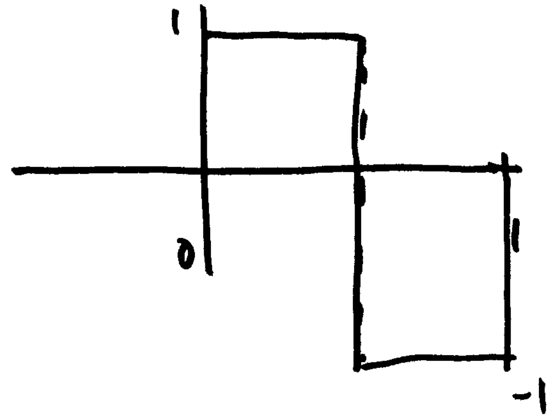
$n=N \to m=2-N$

$$= \sum_{m=2-N}^{1} x(2-m)\, e^{2\pi i(k-1)(1-m)/N}$$

$$= \sum_{m=2-N}^{1} x(m)\, e^{-2\pi i(k-1)(m-1)/N}$$

$$= \sum_{m=1}^{N} x(m)\, e^{-2\pi i(k-1)(m-1)/N} = X(k)$$

wavelet:



Good

small support

good localization in time

fast algorithm

orthonormal.

GOAL: Design new wavelets that have most of good properties of Haar but not all the bad.

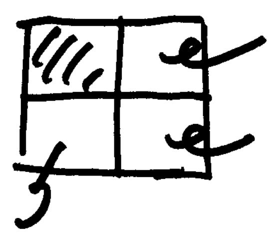
MAIN IDEA: Multiresolution Analysis.

Bad

Discontinuous

Smooth ftns are poorly approximated by partial sums of Haar series.

e.g. Partial reconstructions of images are blocky



Haar coeffs will "find" jump discontinuities but not corners.