Fourier series

$\{e_{n/a}(x)\}_{n\in\mathbb{Z}}$ $\quad e_{n/a}(x) = e^{2\pi i n x/a}$

on $[0,a]$

want to write $f = \frac{1}{a}\sum_n \underbrace{\langle f, e_{n/a}\rangle}\, e_{n/a}$

coefficients
convergence
✓ completeness / uniqueness

$\{\cos(2\pi n x/a), \sin(2\pi n x/a), 1\}_{n=1}^{\infty}$

odd $\{\sin(2\pi n x/a)\}_{n=1}^{\infty} = \{s_n(x)\}_{n=1}^{\infty}$

Take $f \to$ write $\frac{1}{a}\sum_n \langle f, s_n\rangle s_n(x) =$ odd part of f

not complete.

General orthonormal systems $\{g_n\}_{n=1}^{\infty}$
Bessel's inequality
Best Approximation Lemma
$\mathrm{Span}\{g_n\}$

e.g. $\{g_n\}_{n=\phi}^{\infty} = \{x^{n-1}\}_{n=1}^{\infty} = \{1, x, x^2, x^3, \dots\}$

$\text{Span}\{g_n\}$ = set of all polynomials

e.g. $\{g_n\} = \{x^{n-1}\}$ Infinite sums look like:

$\sum_{n=0}^{\infty} a(n) x^n$ power series

e.g. $\sum_{n=0}^{\infty} n! x^n$ converges only at $x=0$

$\sum_{n=0}^{\infty} 2^{-n} x^n$ converges only if $|x|<2$

Pf: ($\Leftarrow$) If $f = \sum_n \langle f, g_n \rangle g_n$ then

this means that

$$f = \lim_{N\to\infty} \sum_{n\geq 1}^{N} \langle f, g_n \rangle g_n$$

Let $f_N = \sum_{n\geq 1}^{N} \langle f, g_n \rangle g_n$. Then $f_N \in \operatorname{span}\{g_n\}$

and $f_N \to f$. $\therefore f \in \overline{\operatorname{span}\{g_n\}}$

($\Rightarrow$) If $f \in \overline{\operatorname{span}\{g_n\}}$ then for every $\varepsilon > 0$ there is a $g \in \operatorname{span}\{g_n\}$ such that $\|f-g\|_2 < \varepsilon$. Let $g = \sum_{n=1}^{N_0} a(n) g_n$

By Lemma,

$$\left\| f - \sum_{n\geq 1}^{N_0} \langle f, g_n \rangle g_n \right\|_2^2 = \|f-g\|_2^2 + \sum_{n\geq 1}^{N_0} |a(n) - \langle f, g_n \rangle|^2$$

$$\therefore \left\| f - \sum_{n\geq 1}^{N_0} \langle f, g_n \rangle g_n \right\|_2 \leq \|f-g\|_2 < \varepsilon.$$

But if $N \geq N_0$

$$\left\| f - \sum_{n\geq 1}^{N} \langle f, g_n \rangle g_n \right\|_2 \leq \left\| f - \sum_{n\geq 1}^{N_0} \langle f, g_n \rangle g_n \right\|_2$$

But this means $f = \sum_{n\geq 1}^{\infty} \langle f, g_n \rangle g_n$. ∎

Pf: (a) $\Leftrightarrow$ (b) Follows from previous Thm.

(a) $\Rightarrow$ (c) obvious since $C_c^0 \subseteq L^2$

(c) $\Rightarrow$ (a) Fact: Given $f \in L^2(I)$ and $\varepsilon > 0$ there is a $g \in C_c^0(I)$ such that $\|f-g\|_2 < \varepsilon$

In this case, $g = \sum_n \langle g, g_n \rangle g_n$ so for some N, $\|g - \sum_{n=1}^{N} \langle g, g_n \rangle g_n\|_2 < \varepsilon$. Therefore

$$\|f - \sum_{n=1}^{N} \langle g, g_n \rangle g_n\|_2 = \|f - g + g - \sum_{n=1}^{N} \langle g, g_n \rangle g_n\|_2$$

$$\leq \|f-g\|_2 + \|g - \sum_{1}^{N} \langle g, g_n \rangle g_n\|_2 < 2\varepsilon$$

$\therefore f \in \overline{\text{span}}\{g_n\}$

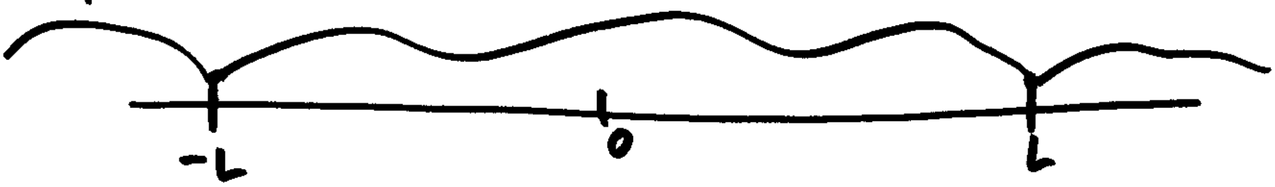
(c) $\Leftrightarrow$ (d) Follows from the identity

$$\|f - \sum_{n=1}^{N} \langle f, g_n \rangle g_n\|_2^2 = \|f\|_2^2 - \sum_{n=1}^{N} |\langle f, g_n \rangle|^2$$

$$\therefore f = \sum_{n=1}^{\infty} \langle f, g_n \rangle g_n \Leftrightarrow \|f\|_2^2 = \sum_{n=1}^{\infty} |\langle f, g_n \rangle|^2$$

Motivation

Spse $f \in L^2(-L,L)$ some $L>0$



Can write

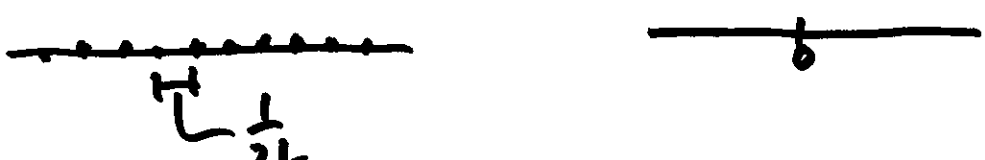
$$f(x) = \frac{1}{2L}\sum_n \langle f, e_{n/2L}\rangle e_{n/2L}(x)$$

$$= \frac{1}{2L}\sum_n \langle f, e_{n/2L}\rangle e^{2\pi i x\left(\frac{n}{2L}\right)}$$

$$\langle f, e_{\frac{n}{2L}}\rangle = \int_{-L}^{L} f(t)\, e^{-2\pi i t(n/2L)}\,dt$$

Let $L \to \infty$ (formally), and let $\langle f, e_{\frac{n}{2L}}\rangle = \hat{f}\left(\frac{n}{2L}\right)$

$$\left\{\frac{n}{2L}\right\}_{n\in\mathbb{Z}} \longrightarrow \mathbb{R}$$



$$\hat{f}\left(\frac{n}{2L}\right) \longrightarrow \hat{f}(\gamma)$$

$$\int_{-L}^{L} f(t)\, e^{-2\pi i t(n/2L)}\,dt \longrightarrow \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t\gamma}\,dt$$

$$\sum_n \hat{f}\left(\frac{n}{2L}\right) e^{2\pi i t\left(\frac{n}{2L}\right)}\frac{1}{2L} \longrightarrow \int_{-\infty}^{\infty} \hat{f}(\gamma)\, e^{2\pi i t\gamma}\,d\gamma$$

$$\| $$

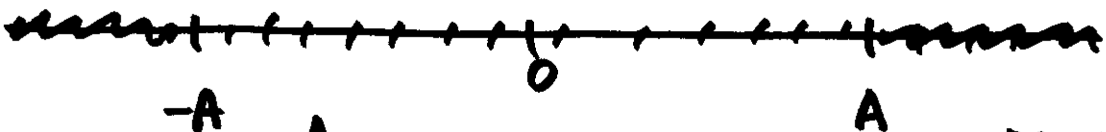
$$f(x)$$

Idea of proof:

$$|\hat{f}(\gamma_1)-\hat{f}(\gamma_2)| = \left|\int_{-\infty}^{\infty} f(x)\,e^{-2\pi i\gamma_1 x}\,dx - \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i\gamma_2 x}\,dx\right|$$

$$= \left|\int_{-\infty}^{\infty} f(x)\,e^{-2\pi i\gamma_2 x}\left(e^{-2\pi i(\gamma_1-\gamma_2)x}-1\right)dx\right|$$

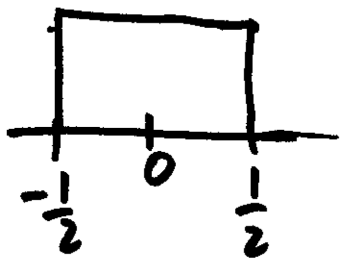
$$\le \int_{-\infty}^{\infty} |f(x)|\,\left|e^{-2\pi i(\gamma_1-\gamma_2)x}-1\right|dx$$



$\int_{-A}^{A} |\ -\ |$ small since $\left|e^{-2\pi i(\gamma_1-\gamma_2)x}-1\right|$ small.

$\int_{|x|>A} |\ -\ |$ small since $\int_{|x|>A} |f|$ is small.

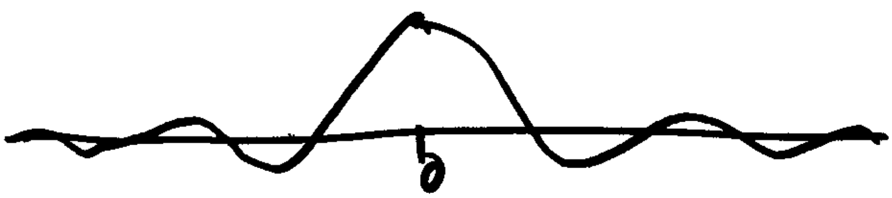
eg: $f(x) =$



$= \chi_{[-1/2,1/2]}(x)$.

$$\hat{f}(\gamma) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \gamma t}\, dt = \frac{1}{-2\pi i \gamma}\left(e^{-\pi i \gamma} - e^{\pi i \gamma}\right)$$

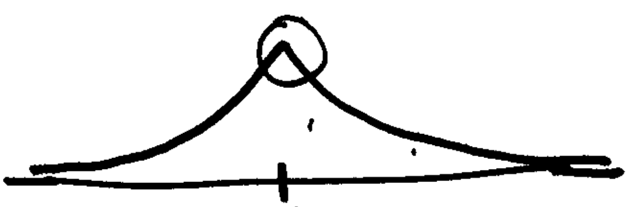
$$= \frac{1}{\pi \gamma}\left(\frac{e^{\pi i \gamma} - e^{-\pi i \gamma}}{2i}\right) = \frac{\sin \pi \gamma}{\pi \gamma}$$



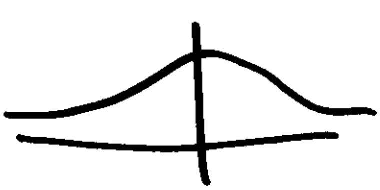
decay like $\frac{1}{|\gamma|}$

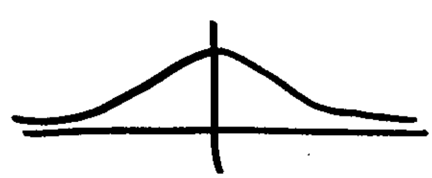
$f \in L^1(\mathbb{R}) \quad \hat{f} \notin L^1(\mathbb{R})$

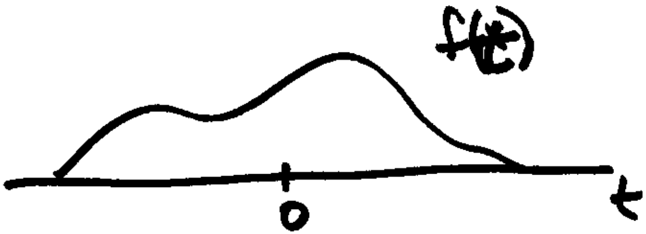
eg. $f(x) = e^{-|x|}$ $\qquad \hat{f}(\gamma) = \frac{1}{1+\gamma^2}$

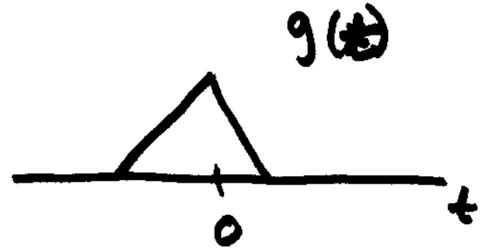


e.g. $f(x) = e^{-\pi x^2}$ $\qquad \hat{f}(\gamma) = e^{-\pi \gamma^2}$



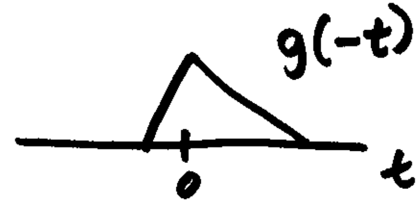




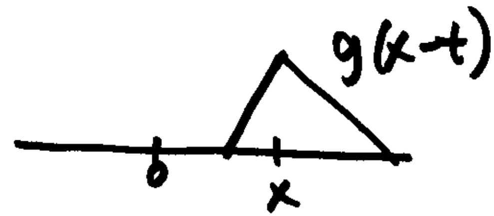


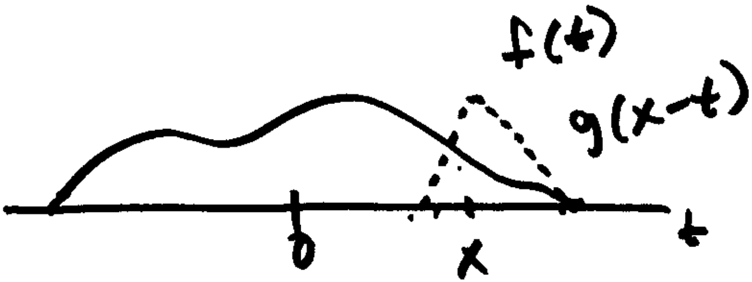
$f*g(x)$:

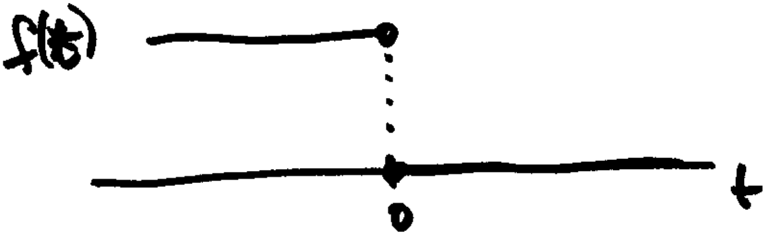
$g(x-t)$: $g(-t)$

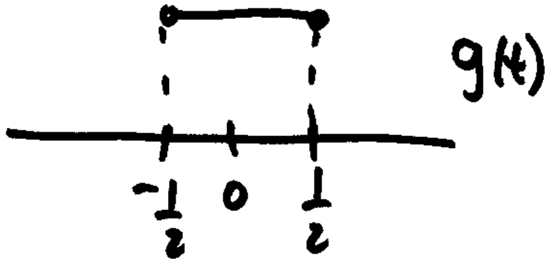


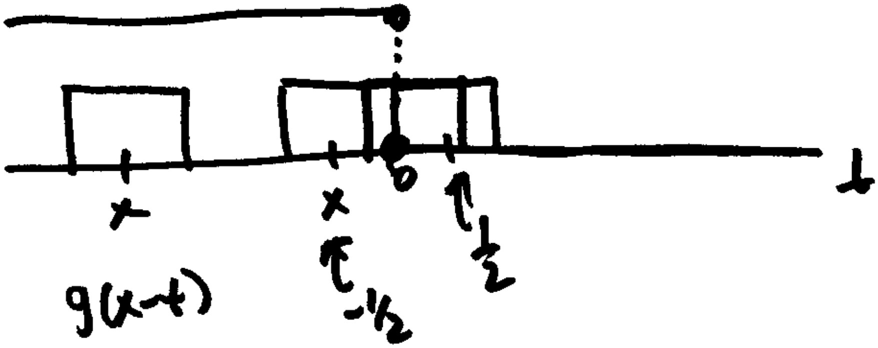
$$g(-(t-x)) = g(x-t)$$

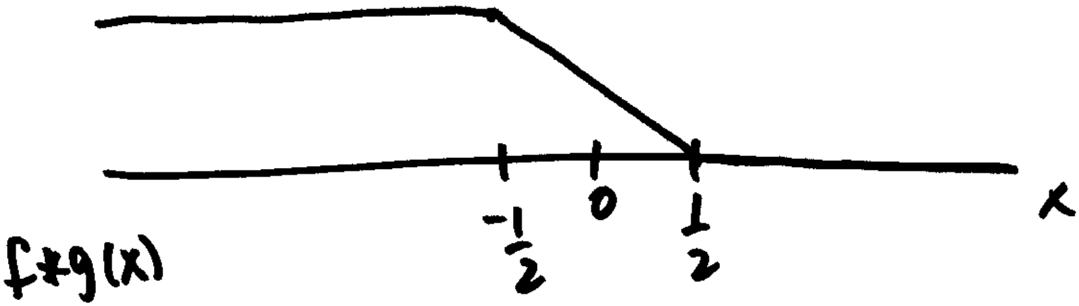





f(t)
0
t


g(t)
-1/2
0
1/2


x
x
0
-1/2
1/2
t
g(x-t)


-1/2
0
1/2
x
f*g(x)

Pf: Say $f \in L^{\infty}$, $g \in L^1$

$$|f*g(x_1) - f*g(x_2)| = \left| \int_{-\infty}^{\infty} f(t)\, g(x_1-t)\,dt - \int_{-\infty}^{\infty} f(t)\, g(x_2-t)\,dt \right|$$

$$= \left| \int_{-\infty}^{\infty} f(t)\big(g(x_1-t) - g(x_2-t)\big)\,dt \right|$$

$$\le \int_{-\infty}^{\infty} |f(t)|\,|g(x_1-t) - g(x_2-t)|\,dt$$

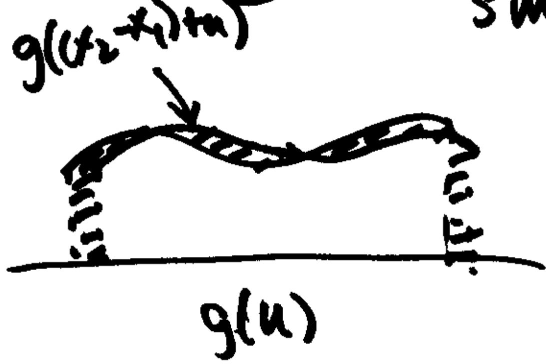
$$\le \|f\|_{\infty} \int_{-\infty}^{\infty} |g(x_1-t) - g(x_2-t)|\,dt$$

$u = x_1 - t$

$du = -dt$

$x_2 - t = x_2 - x_1 + u$

$$= \|f\|_{\infty} \underbrace{\int_{-\infty}^{\infty} |g(u) - g((x_2-x_1)+u)|\,du}_{\text{small by continuity of translation in } L^1 \text{ or } L^2 \text{ norm.}}$$



Pf: $\widehat{f*g}(\gamma) = \int_{-\infty}^{\infty} f*g(x)\, e^{-2\pi i x\gamma}\, dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{-2\pi i x\gamma}\, dt\, dx$$

$$= \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(x-t)\, e^{-2\pi i x\gamma}\, dx\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t\gamma} \underbrace{\int_{-\infty}^{\infty} g(x-t)\, e^{-2\pi i (x-t)\gamma}\, dx}_{\substack{\int_{-\infty}^{\infty} g(u)\, e^{-2\pi i u t}\, du \\ \hat{g}(\gamma)}}\, dt}_{\hat{f}(\gamma)}$$

$$= \hat{f}(\gamma)\, \hat{g}(\gamma).$$

Pf. (Plancherel)

Define $\tilde{f}(x) = \overline{f(-x)}$

Then $$\hat{\tilde{f}}(\gamma) = \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-2\pi i x\gamma}\, dx$$

$$= \overline{\int_{-\infty}^{\infty} f(-x)\, e^{2\pi i x\gamma}\, dx}$$

$$= \overline{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\gamma}\, dx} = \overline{\hat{f}(\gamma)}$$

$$f * \tilde{f}(x) = \int_{-\infty}^{\infty} f(t)\, \tilde{f}(x-t)\, dt = \int_{-\infty}^{\infty} f(t)\, \overline{f(t-x)}\, dt$$

$$\widehat{f * \tilde{f}}(\gamma) = \hat{f}(\gamma)\, \hat{\tilde{f}}(\gamma) = |\hat{f}(\gamma)|^2$$

By Fourier inversion:

$$f * \tilde{f}(x) = \int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2 e^{2\pi i x\gamma}\, d\gamma$$ Set $x=0$.

$$f * \tilde{f}(0) = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2\, d\gamma \qquad \blacksquare$$

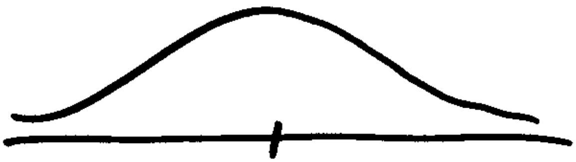
$\underline{\text{Pf}}$:

$$\widehat{xf}(\gamma) = \int_{-\infty}^{\infty} x f(x) e^{-2\pi i x\gamma} \, dx$$

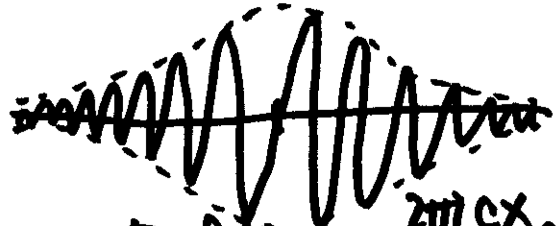
$$= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} f(x) \underbrace{\left(-2\pi i x \, e^{-2\pi i x\gamma}\right)}_{\frac{d}{d\gamma} e^{-2\pi i x\gamma}} dx$$

$$= \frac{-1}{2\pi i} \frac{d}{d\gamma} \left(\int_{-\infty}^{\infty} f(x) e^{-2\pi i x\gamma} dx \right)$$

$$= \frac{-1}{2\pi i} \frac{d}{d\gamma} \hat{f}(\gamma).$$



f



$E_c f(x) = e^{2\pi i c x} f(x)$

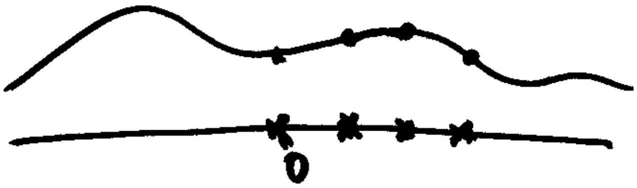
$c > 0$ large

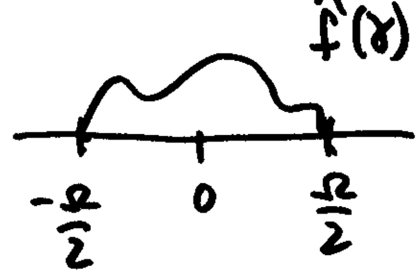
(b) $$\widehat{T_b f}(\gamma) = \int_{-\infty}^{\infty} f(x-b)\, e^{-2\pi i x \gamma}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i (x+b)\gamma}\, dx$$

$$= e^{-2\pi i b \gamma} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \gamma}\, dx$$

$$= E_{-b} \hat{f}(\gamma)$$





f

$$e^{2\pi i \Omega/2\,\gamma} = e^{\pi i \Omega \gamma}$$

period $\frac{2}{\Omega}$

f band limited
says f is "slowly varying"

Idea of Pf: $\hat{f}(\gamma) = 0$ outside $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$

Can write $\hat{f}(\gamma) = \frac{1}{\Omega}\sum_n \langle \hat{f}, e_{\frac{n}{\Omega}} \rangle e_{\frac{n}{\Omega}}(\gamma)$

$$\langle \hat{f}, e_{\frac{n}{\Omega}} \rangle = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{-2\pi i n/\Omega\,\gamma}\, d\gamma = f\left(\tfrac{-n}{\Omega}\right)$$

$$\hat{f}(\gamma) = \frac{1}{\Omega} \sum_n f\left(\tfrac{n}{\Omega}\right) e^{-2\pi i n/\Omega\,\gamma}$$

$$f(x) = \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \hat{f}(\gamma) e^{2\pi i x \gamma}\, d\gamma = \sum_n f\left(\tfrac{n}{\Omega}\right) \frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{2\pi i (x - \frac{n}{\Omega})\gamma}\, d\gamma$$

$$= \sum_n f\left(\tfrac{n}{\Omega}\right) \frac{\sin \pi\Omega(x - \frac{n}{\Omega})}{\pi\Omega(x - \frac{n}{\Omega})}$$