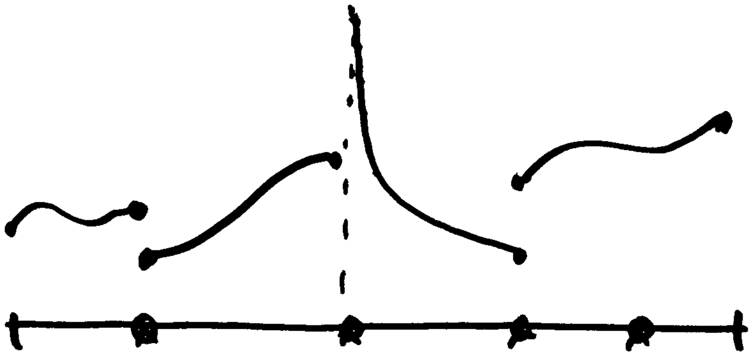
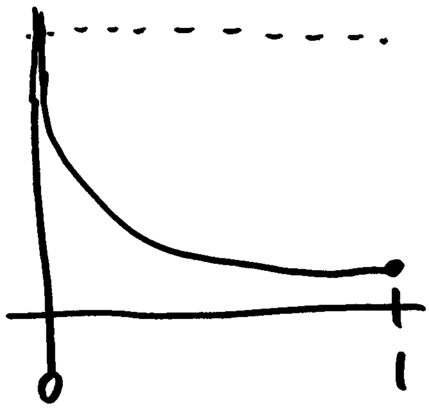


piecewise continuous

I

e.g.



$\frac{1}{x}$ on $(0,1]$

not in $L^\infty(I)$

We say $L^\infty(I) = \{f : f \text{ is p.c. on } I \text{ and } \|f\|_\infty < \infty\}$.

---

e.g. $f \in L^\infty(I)$ I is finite then $f \in L^1(I)$.

$$\int_I |f(x)|dx \leq M \, len(I) < \infty$$

I

Area $= \int_I |f(x)|dx$

e.g. $f(x) = x^{-\alpha}$ on $(0,1]$ $\quad \underline{0<\alpha<1}$

$$\int_0^1 |f(x)|dx = \int_0^1 x^{-\alpha}dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-\alpha}dx = \lim \frac{1}{1-\alpha} x^{1-\alpha}\Big|_\varepsilon^1$$

$$= \lim \frac{1}{1-\alpha}(1-\varepsilon^{1-\alpha}) \to \frac{1}{1-\alpha} < \infty$$

<u>Pf</u>: $f \in L^1(\mathbb{R})$ means $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$. That is,

$$\lim_{r\to\infty} \int_{-r}^{r} |f(x)|\,dx = \int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$ In other words

$$\lim_{r\to\infty} \left(\int_{-\infty}^{\infty} |f(x)|\,dx - \int_{-r}^{r} |f(x)|\,dx \right) = 0.$$ But this means

$$\lim_{r\to\infty} \int_{|x|>r} |f(x)|\,dx = 0.$$ So given $\varepsilon > 0$ there is

an $R > 0$ such that $\int_{|x|>R} |f(x)|\,dx < \varepsilon$

Now let $g(x) = f(x)\mathbb{1}_{[-R,R]}(x)$. Then

$$\int_{-\infty}^{\infty} |f(x) - g(x)|\,dx = \int_{|x|>R} |f(x)|\,dx < \varepsilon.$$

$v = (v_1, v_2, \ldots, v_n) \quad |v| = (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$

$\|f\|_2$ is a generalization of Euclidean length.

<u>Pf</u>: Given $f, g \in L^2(I)$, and $t \in \mathbb{R}$

$$0 \leq \int_I (f(x) + t g(x))^2 dx = \int_I |f(x)|^2 + 2t f(x) g(x) + t^2 |g(x)|^2 dx$$

$$= t^2 \int_I |g(x)|^2 dx + 2t \int_I f(x) g(x) dx + \int_I |f(x)|^2 dx$$

for all $t \in \mathbb{R}$

$At^2 + Bt + C \geq 0$ all t

implies $B^2 - 4AC \leq 0$

Here this means

$$4 \left(\int_I fg\right)^2 \leq 4 \int_I |f|^2 \int_I |g|^2$$

$$\therefore \left|\int_I fg\right| \leq \left(\int_I |f|^2\right)^{1/2} \left(\int_I |g|^2\right)^{1/2} \quad \blacksquare$$

Minkowski's Inequality is also called the triangle inequality for L^2-norm.

<u>Pf</u>: $\|f+g\|_2^2 = \int_I |f+g|^2$

$$= \int_I |f|^2 + 2fg + |g|^2$$

$$\le \int_I |f|^2 + 2\left|\int_I fg\right| + \int_I |g|^2$$

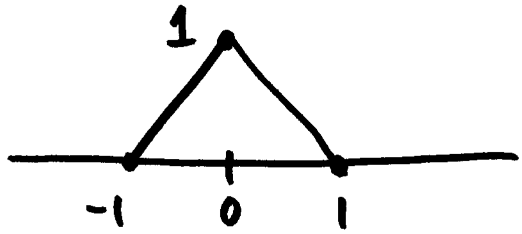
$$\le \int_I |f|^2 + 2\left(\int_I |f|^2\right)^{1/2}\left(\int_I |g|^2\right)^{1/2} + \int_I |g|^2$$

$$= \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2$$

$$= (\|f\|_2 + \|g\|_2)^2 \qquad \blacksquare$$

$f(x)$ is <u>compactly</u> <u>supported</u> if there is an interval $[a,b]$ with a,b finite such that $f(x)=0$ if $x \notin [a,b]$.

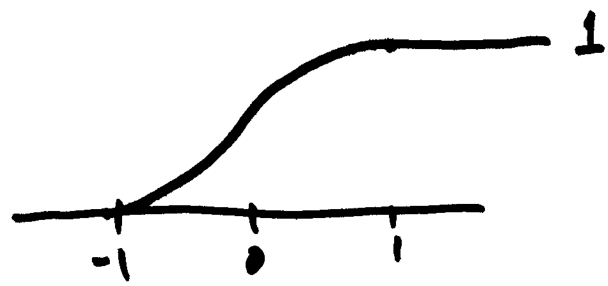
<u>e.g.</u>



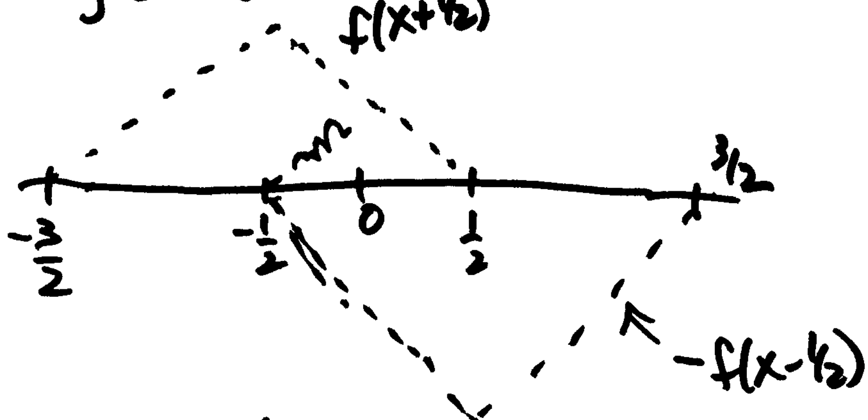
$f(x)$

$f(x) \in C^0_c(\mathbb{R})$

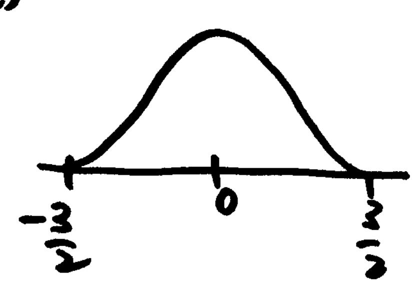
$g(x) = \int_{-\infty}^{x} f(t)\,dt$:



$g \in C^1(\mathbb{R})$



$B_1(x) = \int_{-\infty}^{x} f(t+1/2) - f(t-1/2)\,dt$:



$B_1(x) \in C^1_c(\mathbb{R})$

B_1 is an example of a box spline function

Claim: $L^2(I) \subseteq L^1(I)$ if I is finite

Pf: Let $f \in L^2(I)$ then

$$\int_I |f(x)|\,dx = \int_I |f(x)| \cdot (1)\,dx \leq \left(\int_I |f(x)|^2 dx\right)^{1/2} \left(\int_I 1^2 dx\right)^{1/2}$$

$$= \|f\|_2 \sqrt{\text{len}(I)} < \infty$$

Claim: $L^1(I) \not\subseteq L^2(I)$ if I finite

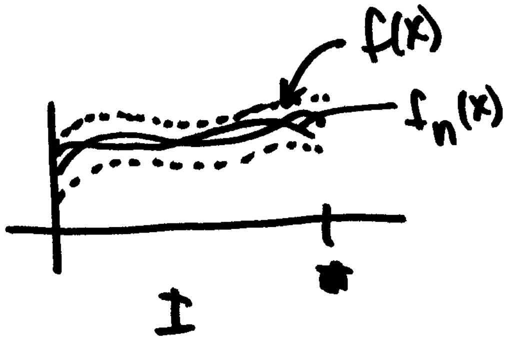
Pf: Let $f(x) = x^{-\alpha}$ on $(0,1]$

$\frac{1}{2} < \alpha < 1$. Then $\int_I |f(x)|\,dx < \infty$

$$\int_0^1 |f(x)|^2 dx = \int_0^1 x^{-2\alpha} dx = \infty$$

since $2\alpha > 1$.

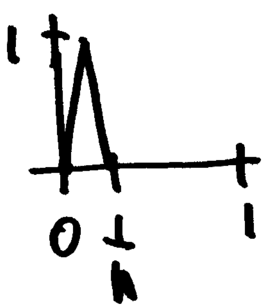
uniform convergence:



$f_n \to f$ in L^∞ means

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

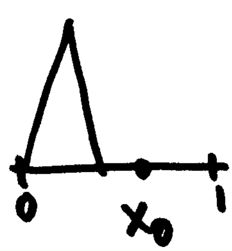
Pointwise convergence $\not\Rightarrow$ uniform convergence.



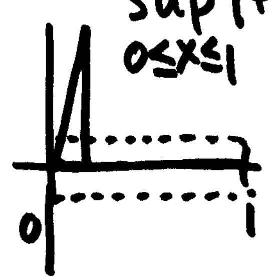
$f_n(x)$ on $[0,1]$.

$f_n \to 0$ pointwise

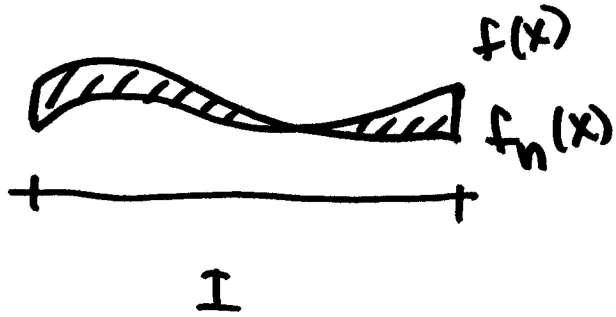
but $f_n \not\to 0$ in L^∞



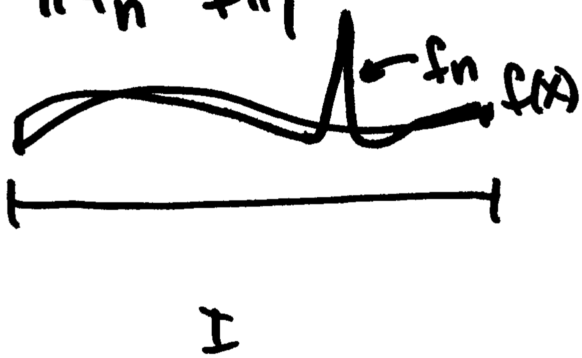
$$\sup_{0\le x\le 1} |f_n(x) - f(x)| = 1$$



L^1 convergence.

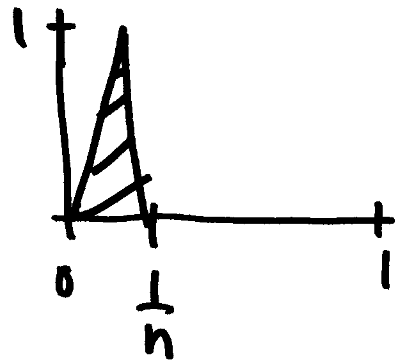


$\|f_n - f\|_1$ = area between curves.



Claim: L^1 convergence $\not\Rightarrow$ L^∞ convergence.

f_n:



$f_n \to 0$ in L^1

$$\int_0^1 |f_n(x) - 0|\,dx = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

L^2 convergence $\not\Rightarrow$ L^∞ convergence.

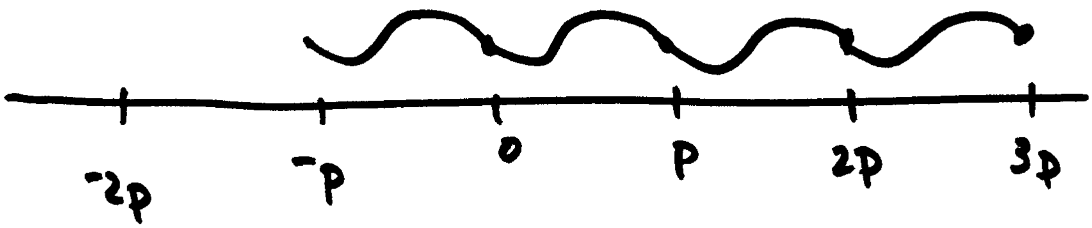
L^2 convergence $\Rightarrow$ L^1 convergence if I is finite

Finite interval

$$\begin{tikzcd}
 & \text{pointwise} & \\
L^1 & & L^2 \arrow[ll, Rightarrow] \\
 & L^\infty \arrow[uu, Rightarrow] \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Arbitrary interval

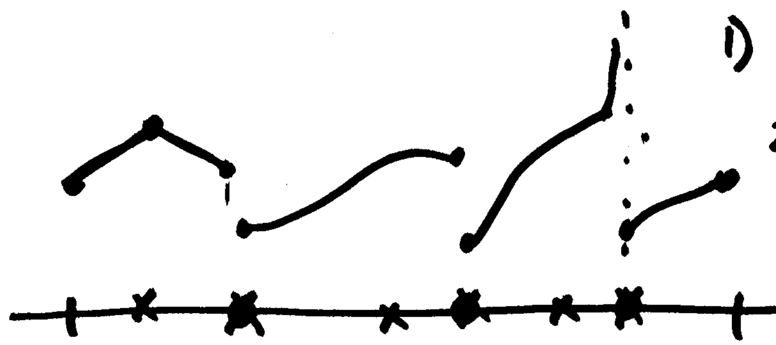
$$\begin{tikzcd}
 & \text{pointwise} & \\
L^1 & & L^2 \\
 & L^\infty \arrow[uu, Rightarrow] &
\end{tikzcd}$$



$$e^{2\pi i n(x+a)/a} = e^{2\pi i n x/a}\, e^{2\pi i n a/a}$$
$$= e^{2\pi i n x/a}\, \underbrace{e^{2\pi i n}}_{=1}$$

---

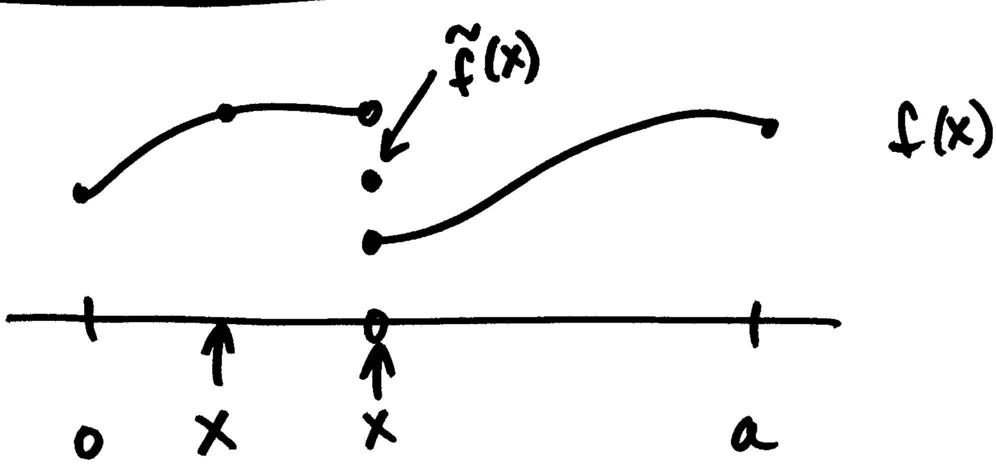
piecewise differentiable



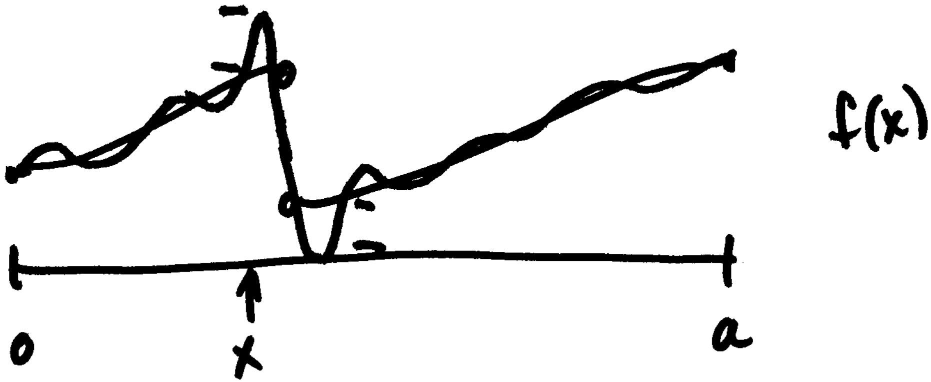
I

1) piecewise cont.

2) f' exists except for finitely many pts. and is piecewise continuous.



$\tilde{f}(x) = f(x)$ if f is cont. at x.



$S_N(x) \not\to f(x)$ uniformly
if $f(x)$ has a jump.

If $f \in L^2(I)$ and $\{g_n\} \subseteq L^2(I)$ then
by Cauchy-Schwarz

$$|\langle f, g_n \rangle| = \left| \int_I f(x)\, \overline{g(x)}\, dx \right| \le \|f\|_2 \, \|g_n\|_2 < \infty$$

$\underline{\text{Pf}}$: $\left\| f - \sum_{n=1}^{N} \langle f, g_n \rangle g_n \right\|_2^2$

$$= \int_I \left(f(x) - \sum_1^N \langle f, g_n \rangle g_n(x) \right) \left(\overline{f(x)} - \sum_1^N \overline{\langle f, g_n \rangle}\, \overline{g_n(x)} \right) dx$$

$$= \int_I |f(x)|^2 dx - \sum_{n=1}^{N} \langle f, g_n \rangle \underbrace{\int_I g_n(x) \overline{f(x)} dx}_{\overline{\langle f, g_n \rangle}}$$

$$- \sum_{n=1}^{N} \overline{\langle f, g_n \rangle} \underbrace{\int_I f(x) \overline{g_n(x)} dx}_{\langle f, g_n \rangle}$$

$$+ \sum_{n=1}^{N} \sum_{m=1}^{N} \langle f, g_n \rangle \overline{\langle f, g_m \rangle} \underbrace{\int_I g_n(x) \overline{g_m(x)} dx}_{\langle g_n, g_m \rangle}$$

$$= \|f\|_2^2 - 2 \sum_{n=1}^{N} |\langle f, g_n \rangle|^2 + \sum_{n=1}^{N} |\langle f, g_n \rangle|^2$$

$$= \|f\|_2^2 - \sum_{n=1}^{N} |\langle f, g_n \rangle|^2 \quad \blacksquare$$

$\underline{\text{Pf}}$ (Bessel's Inequality)

$$0 \le \left\| f - \sum_1^N \langle f, g_n \rangle g_n \right\|_2^2 = \|f\|_2^2 - \sum_1^N |\langle f, g_n \rangle|^2$$

$\therefore \sum_1^N |\langle f, g_n \rangle|^2 \le \|f\|_2^2$ for all N.

$$\therefore \sum_1^\infty |\langle f, g_n \rangle|^2 \le \|f\|_2^2$$

$$\sum_{n=1}^{N} |a(n) - \langle f, g_n \rangle|^2 \geq 0$$

$$\therefore \left\| f - \sum_{n=1}^{N} \langle f, g_n \rangle g_n \right\|_2^2 \leq \left\| f - \sum_{n=1}^{N} a(n) g_n \right\|_2^2$$

previous lecture → $\tilde{e}(x)$ $\qquad\qquad$ $e(x)$

<u>Pf</u>: $\left\| f - \sum_{n=1}^{N} a(n) g_n \right\|_2^2$

$$= \|f\|_2^2 - \sum_{1}^{N} a(n) \overline{\langle f, g_n \rangle} - \sum_{1}^{N} \overline{a(n)} \langle f, g_n \rangle + \sum_{n=1}^{N} \sum_{m=1}^{N} a(n) \overline{a(m)} \langle g_n, g_m \rangle$$

$$= \|f\|_2^2 - \sum_{1}^{N} a(n) \overline{\langle f, g_n \rangle} - \sum_{1}^{N} \overline{a(n)} \langle f, g_n \rangle + \sum_{1}^{N} |a(n)|^2$$

$$= \|f\|_2^2 - \sum_{1}^{N} |\langle f, g_n \rangle|^2 + \left(\sum_{1}^{N} |\langle f, g_n \rangle|^2 - a(n) \overline{\langle f, g_n \rangle} - \overline{a(n)} \langle f, g_n \rangle + |a(n)|^2 \right)$$

$$= \|f\|_2^2 - \sum_{1}^{N} |\langle f, g_n \rangle|^2 + \sum_{1}^{N} |a(n) - \langle f, g_n \rangle|^2$$

$$= \left\| f - \sum_{1}^{N} \langle f, g_n \rangle g_n \right\|_2^2 + \sum_{1}^{N} |a(n) - \langle f, g_n \rangle|^2 \qquad \square$$