Solution to 671 Midterm Problem 2

(a) Let $n, m \geq 0$ and compute

$$\int_0^{\pi} \sin(n+\tfrac{1}{2})x \, \sin(m+\tfrac{1}{2})x \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+\tfrac{1}{2})x \, \sin(m+\tfrac{1}{2})x \, dx \quad \text{since integrand is even}$$

$$= \frac{1}{2}\left(\frac{-1}{4}\right)\int_{-\pi}^{\pi} \left(e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}\right)\left(e^{i(m+\frac{1}{2})x} - e^{-i(m+\frac{1}{2})x}\right) dx$$

$$= \frac{-1}{8}\int_{-\pi}^{\pi} \left(e^{i(n+m+1)x} - e^{i(m-n)x} - e^{i(n-m)x} + e^{-i(n+m+1)x}\right) dx$$

$$= \frac{-1}{8}\begin{cases} -4\pi & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases} = \begin{cases} \frac{\pi}{2} & \text{if } n=m \\ 0 & \text{if } n \neq m. \end{cases}$$

Hence $\{\sin(n+\tfrac{1}{2})x\}_{n=0}^{\infty}$ is an orthogonal system

(b) Now, if $f(x) \sim \sum_{n=0}^{\infty} c_n \sin(n+1/2)x$

we must have $c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(n+1/2)x\,dx$

It will be enough to show that for all f R.I. on $[0,\pi]$ that

$$\sum_{n=0}^{\infty} |c_n|^2 = (2/\pi)\int_0^{\pi} |f(x)|^2\,dx.$$

Let $g(x) = (f(x)+f(2\pi-x))e^{-ix/2}$ $x\in[0,2\pi]$

Then for every $n\in\mathbb{Z}$,

$$\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(f(x)+f(2\pi-x))e^{-ix/2}e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(x)e^{-i(n+1/2)x}\,dx + \int_{\pi}^{2\pi} f(2\pi-x)e^{-i(n+1/2)x}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(x)e^{-i(n+1/2)x}\,dx + \int_0^{\pi} f(t)e^{-i(n+1/2)(2\pi-t)}\,dt\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(x)e^{-i(n+1/2)x}\,dx + \int_0^{\pi} f(t)e^{-2\pi i n}e^{-i\pi}e^{i(n+1/2)t}\,dt\right]$$

$$= \frac{1}{2\pi} \int_0^{\pi} f(x) \left(e^{-i(n+1/2)x} - e^{i(n+1/2)x} \right) dx$$

$$= \frac{-i}{\pi} \int_0^{\pi} f(x) \sin (n+1/2) x \, dx$$

If $n \geq 0$ then $\hat{g}(n) = \frac{-i}{2} c_n$ and if $n < 0$

then

$$\hat{g}(n) = \frac{-i}{\pi} \int_0^{\pi} f(x) \sin(-(-n - 1/2) x) \, dx$$

$$= \frac{i}{\pi} \int_0^{\pi} f(x) \sin(((|n| - 1) + 1/2) x \, dx = \frac{i}{2} c_{|n|-1}$$

Also $\displaystyle\int_0^{2\pi} |g(x)|^2 dx = \int_0^{\pi} |f(x)|^2 dx + \int_{\pi}^{2\pi} |f(2\pi - x)|^2 dx$

$$= 2 \int_0^{\pi} |f(x)|^2 dx$$

Finally,

$$\sum_{n=0}^{\infty} |c_n|^2 = \frac{1}{2} \left(\sum_{n=0}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_{n-1}|^2 \right)$$

$$= \frac{1}{2} \left(\sum_{n=0}^{\infty} 4 |\hat{g}(n)|^2 + \sum_{n=-\infty}^{-1} 4 |\hat{g}(n)|^2 \right)$$

$$= 2 \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 \overset{(*)}{=} \frac{1}{\pi}\int_0^{2\pi} |g(x)|^2 dx = \frac{2}{\pi}\int_0^{\pi} |f(x)|^2 dx$$

(*) Note that if

$$\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(x) e^{-inx}\, dx = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} g(x) \frac{e^{-inx}}{\sqrt{2\pi}}\, dx$$

Hence

$$\sum_{n\in\mathbb{Z}} |\sqrt{2\pi}\, \hat{g}(n)|^2 = \int_0^{2\pi} |g(x)|^2 dx$$ by Bessel's inequality

since $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}$ is an ortho<u>normal</u> system

and so $$\sum_{n\in\mathbb{Z}} |\hat{g}(n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2 dx.$$