MATH 671 – HOMEWORK #6
DUE 29 OCTOBER 2012

Exercise 1. For each pair of integers (j, k) define the *dyadic interval* $I_{j,k}$ by

$$I_{j,k} = [2^{-j}k, 2^{-j}(k + 1)],$$

and the *Haar function* $h_{j,k}(x)$ by

$$h_{j,k} = 2^{j/2}(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}})$$

where for any interval I, $\chi_I(x) = 1$ if $x \in I$ and 0 if $x \notin I$. Prove the following.

(a) For all (j, k), $I_{j,k} = I_{j+1,2k} \cup I_{j+1,2k+1}$.

(b) The system $\{h_{j,k}\}_{j,k\in\mathbf{Z}}$ is an orthonormal system with respect to the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$. (Hint: You may use the result of part (a) to show this.)

Exercise 2. Consider the *Legendre polynomials* $\{L_n\}_{n=0}^{\infty}$ defined by the formula at the beginning of Problem 2, p. 95 of Stein and Shakarchi. Prove the following.

(a) Each $L_n(x)$ is a polynomial of degree n. (Hint: Use induction.)

(b) The system $\{L_n\}_{n=0}^{\infty}$ is an orthogonal system in the space of Riemann integrable functions on $[-1, 1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$. (Hint: Use integration by parts in the strategy of Problem 2(a) of Stein and Shakarchi.)

(c) For each n, the monomial x^n is in the linear span of $\{L_m\}_{m=0}^{n}$. (Hint: Use induction once again.)

Exercise 3. Exercise 7, p. 123 in Stein and Shakarchi.