9.2/9.3 Continuous Functions.

A. Definition of Continuity.

Definition. Let $\mathbb{f}: D \to \mathbb{E}^m$, where $D = D_{\mathbb{f}} \subseteq \mathbb{E}^n$. Let $\mathbb{a} \in D$. We say that $\mathbb{f}$ is *continuous at* $\mathbb{a}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $\mathbb{x} \in D \cap B(\mathbb{a}, \delta)$, $\|\mathbb{f}(\mathbb{x}) - \mathbb{f}(\mathbb{a})\| < \epsilon$.

We say that $\mathbb{f}$ is *continuous on* D if it is continuous at every point of D and we write $\mathbb{f} \in C(D)$.

Theorem. A function $\mathbb{f}$ is continuous at a limit point $\mathbb{a} \in D_{\mathbb{f}}$ if and only if

$$\lim_{\mathbb{x}\to\mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{f}(\mathbb{a}).$$

Proof.

Definition. A set $E \subseteq \mathbb{E}^n$ is *relatively open in* $D \subseteq \mathbb{E}^n$ if there is an open set U in $\mathbb{E}^n$ such that $E = U \cap D$. E is *relatively closed in* D if there is a closed set $F \subseteq \mathbb{E}^n$ such that $E = F \cap D$.

Remark. (a) Note that for E to be relatively open or relatively closed in D it must be true that $E \subseteq D$.

(b) The concept of *open* and *closed* in a normed vector space is dependent on the space in which the set is assumed to sit. In other words, a set E is not simply open or closed, but is only open or closed *as a subset of some other set*.

(c) For example, the set $[0,1]$ is not open as a subset of $\mathbb{R}$ (the ambient vector space), but is open as a subset of $[0,1]$, that is if we assume that our ambient vector space is $[0,1]$.

Theorem (9.2.3). A subset $E \subseteq D$ is relatively open in D if and only if for every $\mathbb{x} \in E$, there is a $\delta > 0$ such that $B(\mathbb{x}, \delta) \cap D \subseteq E$.

Proof.

Theorem. (9.2.4). Let $\mathbb{f}: D \to \mathbb{E}^n$. Then $\mathbb{f} \in C(D)$ if and only if for every open set U in $\mathbb{E}^n$, the inverse image

$$\mathbb{f}^{-1}(U) = \{\mathbb{x} \in D: \mathbb{f}(\mathbb{x}) \in U\}$$

is relatively open in D.

Remark. (a) Note that it is not necessarily true that the forward image of an open set by a continuous function is open.

(b) Nor is it necessarily true that the forward image of a closed set by a continuous function is closed.

Theorem (9.3.1). Let $\mathbb{f} \in C(D)$, $\mathbb{f}: D \to \mathbb{E}^m$. If D is compact, then $\mathbb{f}(D) = \{\mathbb{f}(\mathbb{x}): \mathbb{x} \in D\}$ is compact.

Theorem (Extreme Value Theorem). If $D \subseteq \mathbb{E}^n$ is compact and if $\mathbb{f} \in C(D)$, is a real-valued function, then $\mathbb{f}$ achieves is maximum and minimum values on D. In other words, there exist points $\mathbb{x}_M$ and $\mathbb{x}_m \in D$ such that $\mathbb{f}(\mathbb{x}) \leq \mathbb{f}(\mathbb{x}_M)$ and $\mathbb{f}(\mathbb{x}) \geq \ \mathbb{x}_m$ for all $\mathbb{x} \in D$.

Theorem (9.3.3) (Open Mapping Theorem.)
Let $\mathbb{f} \in C(D)$, $\mathbb{f}: D \to \mathbb{E}^m$, $\mathbb{f}$ is one-to-one. If D is compact then $\mathbb{f}^{-1}$ is continuous.

Corollary. Under the hypotheses of the Theorem, the forward image of a relatively open set in D is open in $\mathbb{E}^m$.