9.1 Limits of Functions.

A. Definition of limit.

1. Definition. We will consider *vector-valued functions*, $\mathbb{f}: D \to \mathbb{E}^m$, with domain $D = D_{\mathbb{f}} \subseteq \mathbb{E}^n$. We write
   $\mathbb{f}(\mathbb{x}) = \mathbb{f}(x_1, \dots, x_n) = (f_1(\mathbb{x}), f_2(\mathbb{x}), \dots, f_m(\mathbb{x}))$
   where $f_i: D \to \mathbb{R}$ and we usually write
   $f_i(\mathbb{x}) = f_i(x_1, x_2, \dots, x_n)$.

2. Definition. Let a be a limit point (cluster point) of the domain $D_{\mathbb{f}}$ of a function $\mathbb{f}$. Then

$$\lim_{\mathbb{x} \to \mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{L}$$

   if for all $\epsilon > 0$ there is an $\delta > 0$ such that for all $\mathbb{x} \in D_{\mathbb{f}}$, if $0 < \|\mathbb{x} - \mathbb{a}\| < \delta$, then $\|\mathbb{f}(\mathbb{x}) - \mathbb{L}\| < \epsilon$.

3. Remark. If $\mathbb{a}$ is an isolated point of $D_{\mathbb{f}}$ then it does not make sense to talk about
   $\lim_{\mathbb{x} \to \mathbb{a}} \mathbb{f}(\mathbb{x})$.

Theorem 1. (9.1.1) Suppose that $\mathbb{a}$ is a limit point of the domain $D_{\mathbb{f}}$ of the function $\mathbb{f}$. Then the following are equivalent

a. $\lim_{\mathbb{x}\to\mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{L}$.

b. For every sequence $\{\mathbb{x}^{(j)}\} \in D_{\mathbb{f}}$, with $\mathbb{x}^{(j)} \neq \mathbb{a}$ for all j, such that $\mathbb{x}^{(j)} \to \mathbb{a}$, $\lim_{j\to\infty} \mathbb{f}\left(\mathbb{x}^{(j)}\right) = \mathbb{L}$.

Proof.

4. Example. Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x)\sin(y)}{x^2+y^2}$ or prove it does not exist.

5. Example. Find $\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2+2y^4}$ or prove it does not exist.

6. Example. Find $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ or prove it does not exist.