8.3. Compact Sets.

A. The Bolzano-Weierstrass Theorem.

1. Theorem. (Bolzano-Weierstrass) Every bounded sequence of real numbers has a convergent subsequence.

2. Theorem. (Bolzano-Weierstrass) Every bounded sequence $\mathbb{x}^k$ in $\mathbb{R}^n$ has a convergent subsequence.

   Proof:

3. Remarks.

a. We want to characterize the sets that have a property like that described in the B-W Theorem, specifically: *For which sets* A *is it true that any sequence* $\{\mathbb{x}^k\} \subseteq A$ *has a convergent subsequence?*

b. B-W Theorem says that if A is a closed ball, this property holds.

c. It is also possible to think of this property as saying: *Which subsets of* $\mathbb{R}$ *are most like finite sets in a particular sense?*

4. <u>Definition.</u> (8.3.1) An *open cover* of a subset $S \subseteq \mathbb{R}$ is a collection of open sets (possibly infinite) $\mathcal{O} = \{O_\alpha : \alpha \in A\}$ such that $S \subseteq \bigcup_{\alpha \in A} O_\alpha$. The set S is said to be *compact* if every open cover of S has a finite subcover, that is if $\{O_\alpha : \alpha \in A\}$ is an open cover of S then there exist indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$S \subseteq \bigcup_{k=1}^{n} O_{\alpha_k}.$$

5. <u>Examples.</u> (a) An open ball $B(\mathbb{a}, r)$ is not compact.

(b) A finite subset of $\mathbb{R}^n$ is compact.

6. Theorem. A closed ball in $\mathbb{R}^n$ is compact.

7. Claim: Any open cover of a set $A \subseteq \mathbb{R}^n$ admits a *countable* subcover.

   Proof of Claim:

Proof of Theorem:

8. <u>Theorem.</u> (8.3.1) A set $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

<u>Proof.</u>

9. Theorem. Let $A \subseteq \mathbb{R}^n$. Then the following are equivalent.

a. A is compact.
b. A is closed and bounded.
c. Every sequence of points in A has a limit point in A, that is, every sequence in A has a convergent subsequence.

Proof: