8.2. Open Sets and Closed Sets.

A. Open Sets.

1. Definition. The *open ball centered at* $\mathbb{a} \in \mathbb{R}^n$ *with radius* $r > 0$, denoted $B(\mathbb{a}, r)$ is the set

$$B(\mathbb{a}, r) = \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x} - \mathbb{a}\| < r\}.$$

A set $\mathcal{O} \subseteq \mathbb{R}^n$ is *open* if for each $\mathbb{x} \in \mathcal{O}$, there is an $r > 0$ such that

$$B(\mathbb{x}, r) \subseteq \mathcal{O}$$

2. Examples.
   a. The empty set $\emptyset$ is open, and $\mathbb{R}^n$ is open.
   b. Any open ball is an open set.
   c. Any open set can be written as the union of a collection of open balls.

3. Theorem. The union of any collection of open sets is open.

Proof.

4. Theorem. The intersection of a finite number of open sets is open. The infinite intersection of open sets need not be open.

Proof.

B. Closed Sets.

1. Definition. Let $A \subseteq \mathbb{R}^n$. The point $\mathbb{x}$ is a *limit point* of A if for every $\epsilon > 0$, there is a $\mathbb{y} \in A$ such that $0 < \|\mathbb{x} - \mathbb{y}\| < \epsilon$.

2. Remark.
   a. A limit point of a set A need not be an element of A. For example, what are the limit points of $B(\mathbb{0}, 1)$?

   b. Claim. A point $\mathbb{x}$ is a limit point of a set A if and only if for every $\epsilon > 0$, $B(\mathbb{x}, r) \cap A$ contains infinitely many points.

   Proof.

c. <u>Definition.</u> Let $A \subseteq \mathbb{R}$. The point $\mathbb{x}$ is an *isolated point* of A if there exists an $r > 0$ such that $B(\mathbb{x}, r) \cap A = \emptyset$.

d. <u>Claim.</u> Every point of a set $A \subseteq \mathbb{R}$ is either an isolated point of A or a limit point of A.

3. <u>Definition.</u> A set $F \subseteq \mathbb{R}^n$ is said to be *closed* if it contains all of its limit points.

4. Examples.
   a. The empty set $\emptyset$ is closed, and $\mathbb{R}$ is closed. This also shows that it is possible for a set to be both open and closed.
   b. An open ball $B(\mathbb{x}, r)$ is not closed.
   c. Any *closed ball*
   $\overline{B(\mathbb{a}, r)} = \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x} - \mathbb{a}\| \leq r\}$
   is closed.
   d. Any finite set is closed.

5. <u>Theorem.</u> A set is closed if and only if its complement is open.

Proof.

6. Theorem. The union of finitely many closed sets is closed and the intersection of any

number of closed sets is closed. The union of infinitely many closed sets need not be closed.