5.5/5.6. The Weierstrass M-test and Power Series.

<u>Definition.</u> A sequence of functions $f_k(x)$ defined on a domain D converges to the function $f(x)$ pointwise on D if for each $x \in D$ the numerical sequence $f_k(x)$ converges to $f(x)$. The convergence is *uniform* if

$$sup_{x\in D}\,|f_k(x) - f(x)| \to 0$$

As $k \to \infty$. The series $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise (resp. uniformly) on D if the sequence of partial sums $s_n(x) = \sum_{k=1}^{n} f_k(x)$ converges pointwise (resp. uniformly) on D.

<u>Theorem 5.5.2.</u> (Weierstrass M-test)
Let $f_k(x)$ be a sequence of functions defined on a domain D. Let

$$M_n = sup_{x\in D}|f_n(x)| = \|f_n\|_{sup} < \infty$$

If $\sum_{n=1}^{\infty} M_n < \infty$ then the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly on D.

Proof:

Definition (Power Series)
A *power* series centered at a (or with *base* point a) is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

where c_k is a sequence of real coefficients.

Theorem 5.6.1. Given a power series $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, there exists a number $0 \leq R \leq \infty$, called the *radius of convergence* with the property that the series converges absolutely on the interval $|x-a| < R$, absolutely and uniformly on any interval $[\alpha, \beta] \subseteq (a-R, a+R)$, and diverges for $|x-a| > R$.

Proof:

Theorem. (Abel) Let $a < b$. If the power series $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ converges pointwise on $[a, b]$, then $f(x)$ is continuous and converges uniformly on $[a, b]$.

Proof.

Theorem. If $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ has radius of convergence $R > 0$, then f is infinitely differentiable on $(a-R, a+R)$, and $f^{(n)}$ is given by

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)c_k(x-a)^{k-n}$$

which power series also has radius of convergence R.

Proof: