5.3. Products of Series.

Theorem 5.3.2. If x_n and y_n are absolutely summable, then so is the doubly indexed sequence $\{x_j\, y_k\}_{j,k=1}^{\infty}$ and

$$\sum_{j,k=1}^{\infty} x_j\, y_k = \left(\sum_{j=1}^{\infty} x_j\right)\left(\sum_{k=1}^{\infty} y_k\right)$$

Proof:

Definition (Cauchy product)
Given sequences x_j , y_k , define their *Cauchy product* $\{c_l\}_{l=2}^{\infty}$ by

$$c_l = \sum_{j+k=l} x_j y_k = \sum_{j=1}^{l-1} x_j y_{l-j}$$

Corollary. If x_j and y_k are absolutely summable, then

$$\sum_{l=2}^{\infty} c_l = \sum_{j,k=1}^{\infty} x_j y_k = \left(\sum_{j=1}^{\infty} x_j\right)\left(\sum_{k=1}^{\infty} y_k\right)$$

Proof:

Definition. (Iterated Sums) Let $a_{j,k}$ be a doubly indexed sequence. The *iterated sum* of the sequence is defined as the series.

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k}$$

In other words, if we define the sequence c_k by $c_k = \sum_{j=1}^{\infty} a_{j,k}$ then

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} c_k$$

Remark Note that the iterated sum is not a rearrangement of the series $\sum_{j,k=1}^{\infty} a_{j,k}$.

Theorem. If $a_{j,k}$ is absolutely summable then

$$\sum_{j,k=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k}$$

Proof.