5.2. Convergence Tests.

<u>Theorem.</u> A series $\sum_{n=1}^{\infty} x_n$ of nonnegative terms, that is, for which $x_n \geq 0$ for all n, converges if and only if the sequence of partial sums is bounded.

<u>Proof:</u>

Theorem 5.2.2. (Integral Test.)
Suppose f is Riemann integrable on $[1, \infty)$, that is, it is integrable on $[1, b]$ for every $b > 0$, is monotone decreasing and approaches 0 as $x \to \infty$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx < \infty$$

Proof:

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem. (Ratio Test.) Suppose that $x_k > 0$ for all k. If

$$liminf_{k\to\infty} \frac{x_{k+1}}{x_k} > 1$$

then the series $\sum_{n=1}^{\infty} x_n$ diverges. If

$$limsup_{k\to\infty} \frac{x_{k+1}}{x_k} < 1$$

then the series $\sum_{n=1}^{\infty} x_n$ converges.

Proof.

Example 5.19(a). Show that

$$\sum_{k=0}^{\infty} \frac{k!}{k^k}$$

converges.