5.1 Series of Constants (continued).

Definition 5.1.2 (Absolute convergence) Let x_n be a sequence of numbers. The series $\sum_{n=1}^{\infty} x_n$ *converges absolutely* if the series $\sum_{n=1}^{\infty} |x_n|$ converges. In this case, we say that the sequence x_n is *absolutely summable.* A series that is convergent but not absolutely convergent is called *conditionally convergent.*

Theorem 5.1.3. Every absolutely summable sequence is summable.

Proof:

Definition. (Unconditional convergence.)
A sequence y_n is a *rearrangement* of a sequence x_n if there is a bijection $f: \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $y_n = x_{f(n)}$. A series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if every rearrangement y_n of x_n is summable.

Lemma. If $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* then for every rearrangement y_n,

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n$$

Proof:

Theorem. A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if and only if it is unconditionally convergent.

Proof.

Example 5.3. (Geometric series)
Given numbers a and r (not necessarily real), the series $\sum_{k=0}^{\infty} ar^k$ is called a *geometric series* with common ratio r.

Lemma. For any $r \neq 1$,

$$s_n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Proof:

Theorem. The geometric series $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise (if $a \neq 0$).

Proof: