5.1 Series of Constants.

<u>Definition 5.1.1</u> (Convergent series) Let x_n be a sequence of numbers. The *(infinite) series* $\sum_{n=1}^{\infty} x_n$ *converges to* s if the sequence of partial sums, $s_n = \sum_{k=1}^{n} x_k$ converges to s. In this case, we say that the sequence of terms x_n is *summable* and that the corresponding series is *convergent.* Otherwise, we say that the series is *divergent.*

<u>Lemma.</u> (Cauchy criterion.) The series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums, $s_n = \sum_{k=1}^{n} x_k$ is Cauchy, that is, given $\epsilon > 0$, there is an N such that if $n, m \geq N$ then $|s_n - s_{m-1}| = |\sum_{k=m}^{n} x_k| < \epsilon$.

<u>Theorem 5.1.1</u> (n^{th} term test)
If x_n is a summable sequence, then $x_n \to 0$.

<u>Proof.</u>

Theorem: The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof:

Theorem. (Abel's Formula or Summation by Parts.)
Let a_n and b_n be real-valued sequences and let $A_n = \sum_{k=1}^{n} a_k$. Then for all $n > 1$
$$\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)$$

Proof:

Theorem. (Dirichlet's Test)
Let a_n and b_n be real-valued sequences and suppose that $s_n = \sum_{k=1}^{n} a_k$ is a bounded sequence and that $b_n \downarrow 0$ (that is, the sequence b_n is decreasing and converges to 0). Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof:

Theorem. (Alternating Series Test)
Suppose that $x_n \downarrow 0$. Then the (alternating) series $\sum_{n=1}^{\infty}(-1)^n\, x_n$ converges. Moreover, if $s = \sum_{n=1}^{\infty}(-1)^n\, x_n$ then $|s_n - s| \leq x_{n+1}$.

Proof:

Theorem. (Convergence of trigonometric series - Dirichlet)
Suppose that $a_n \downarrow 0$. Then for every $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges.

Proof: