MATH 316 – HOMEWORK 9 – SOLUTIONS

Exercise 10.30.

Solution:

To prove that T is differentiable at $\mathbf{x}$, we must find a $T' \in \mathcal{L}(\mathbf{E}^n, \mathbf{E}^m)$ that satisfies

$$\lim_{\mathbf{h}\to 0} \frac{\|T(\mathbf{x}+\mathbf{h}) - T(\mathbf{x}) - T'(\mathbf{h})\|}{\|h\|} = 0.$$

Since the derivative is unique, once we have identified such a T' we are done. Let $T' = T$. Since T is linear $T(\mathbf{x}+\mathbf{h}) - T(\mathbf{x}) = T(\mathbf{x}) + T(\mathbf{h}) - T(\mathbf{x}) = T(\mathbf{h})$. Therefore, letting $T' = T$,

$$\frac{\|T(\mathbf{x}+\mathbf{h}) - T(\mathbf{x}) - T'(\mathbf{h})\|}{\|h\|} = \frac{\|T(\mathbf{x}) + T(\mathbf{h}) - T(\mathbf{x}) - T(\mathbf{h})\|}{\|h\|} = 0.$$

Therefore the definition of derivative is satisfied for $T' = T$.

2. Exercise 10.38.

Solution:

(a). If $\mathbf{x} \neq 0$ then $f(x_1, x_2) = (x_1 x_2)/(x_1^2 + x_2^2)$ so we can use ordinary rules of differentiation to find $\partial f/\partial x_1$ and $\partial f/\partial x_2$. Doing so, we get

$$\frac{\partial f}{\partial x_1} = \frac{x_2(x_2^2 - x_1^2)}{(x_1^2 + x_2^2)^2}; \quad \frac{\partial f}{\partial x_2} = \frac{x_1(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}.$$

At $\mathbf{x} = 0$ we use the definition to obtain

$$f_{x_1}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and similarly for $f_{x_2}(0,0) = 0$.

(b). f is not differentiable at $(0,0)$ because it is not continuous there. This can be shown by showing that the limit differs when $(0,0)$ is approached for example along the positive x-axis and along the line $y = x$.

3. Exercise 10.39.

Solution:

(a). If $\mathbf{x} \neq 0$ then there is nothing to prove because f is differentiable at each non-zero point so the directional derivative is just the derivative matrix multiplied by the direction vector. At $\mathbf{x} = 0$ we compute At $\mathbf{x} = 0$ we use the definition to obtain

$$D_{\mathbf{v}} f(0,0) = \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{(t^2 v_1^2)(tv_2)}{t(t^4 v_1^4 + t^2 v_2^2)} = \lim_{t\to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} = \frac{v_1^2}{v_2},$$

if $v_2 \neq 0$ and clearly the limit is 0 otherwise.

(b). f is not differentiable at $(0,0)$ because it is not continuous there. This example is nearly identical to Exercise 9.5.