MATH 316 – HOMEWORK 8 – SOLUTIONS

Exercise 10.1.
Solution:

We will show that

$$\sup_{\mathbf{x}\neq 0} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|} = \inf\{C\colon \|L\mathbf{x}\| \leq C\|\mathbf{x}\|, \forall \mathbf{x} \in \mathbf{E}^n\}.$$

Let $A = \sup_{\mathbf{x}\neq 0} \|L\mathbf{x}\|/\|\mathbf{x}\|$. Then A is an upper bound of the set $\{\|L\mathbf{x}\|/\|\mathbf{x}\|, \mathbf{x} \neq 0\}$ and hence for all $\mathbf{x} \neq 0$, $\|L\mathbf{x}\| \leq A\|\mathbf{x}\|$, and clearly the inequality also holds for $\mathbf{x} = 0$. Therefore, $A \in \{C\colon \|L\mathbf{x}\| \leq C\|\mathbf{x}\|, \forall \mathbf{x}\}$ and

$$\inf\{C\colon \|L\mathbf{x}\| \leq C\|\mathbf{x}\|, \forall \mathbf{x} \in \mathbf{E}^n\} \leq A = \sup_{\mathbf{x}\neq 0} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|}.$$

Given $A \in \{C\colon \|L\mathbf{x}\| \leq C\|\mathbf{x}\|, \forall \mathbf{x}\}$, $\|L\mathbf{x}\| \leq A\|\mathbf{x}\|$ for all $\mathbf{x} \neq 0$. Hence A is an upper bound of the set $\{\frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|}, \mathbf{x} \neq 0\}$, and so $\sup_{\mathbf{x}\neq 0} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|} \leq A$. Therefore, taking the infimum over all such A, we have that

$$\sup_{\mathbf{x}\neq 0} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|} \leq \inf\{C\colon \|L\mathbf{x}\| \leq C\|\mathbf{x}\|, \forall \mathbf{x} \in \mathbf{E}^n\}.$$

This last inequality is all that is needed to do the problem as stated.

Exercise 10.4.
Solution:

By the triangle inequality for the Euclidean norm, if $\mathbf{x} \in \mathbf{E}^n$ then

$$\|(L+L')\mathbf{x}\| = \|L\mathbf{x}+L'\mathbf{x}\| \leq \|L\mathbf{x}\|+\|L'\mathbf{x}\| \leq \|L\|\|\mathbf{x}\|+\|L'\|\|\mathbf{x}\| = (\|L\|+\|L'\|)\|\mathbf{x}\|.$$

This means that the number $\|L\|+\|L'\|$ is in the set $\{C\colon \|(L+L')\mathbf{x}\| \leq C\|\mathbf{x}\|, \forall \mathbf{x} \in \mathbf{E}^n\}$. Since $\|L + L'\|$ is the infimum of all such numbers, $\|L + L'\| \leq \|L\| + \|L'\|$.

Exercise 10.11.
Solution:

(a). we must show that the sequence T_K is Cauchy in the space of linear operators on $\mathbf{E}^n$, $\mathcal{L}(\mathbf{E}^n)$. That is, we must show that given $\epsilon > 0$ there is an N such that if n, $m \geq N$ then $\|T_n - T_m\| < \epsilon$ where $\|\cdot\|$ is the operator norm. Since $T_K = \sum_{k=0}^{K} X^k$, then by the triangle inequality and Exercise 10.8(b),

$$\|T_n - T_m\| = \left\| \sum_{k=m+1}^{n} X^k \right\| \leq \sum_{k=m+1}^{n} \|X^k\| \leq \sum_{k=m+1}^{n} \|X\|^k.$$

Since $\|X\| < 1$, the series $\sum \|X\|^k$ converges, and hence there is an N such that if n, $m \geq N$ then $\sum_{k=m+1}^{n} \|X\|^k < \epsilon$. Hence for such n, m, $\|T_n - T_m\| < \epsilon$. By the completeness of $\mathcal{L}(\mathbf{E}^n)$ (Exercise 10.10), there is a $T \in \mathcal{L}(\mathbf{E}^n)$ such that $T_K \to T$, that is, $\|T_K - T\| \to 0$ as $K \to \infty$.

(b). Note that

$$\begin{aligned}
(I - X)T_K - I &= T_K - X\,T_K - I \\
&= \sum_{k=0}^{K} X^k - X \sum_{k=0}^{K} X^k - X^0 \\
&= \sum_{k=0}^{K} X^k - \sum_{k=1}^{K+1} X^k - X^0 \\
&= \sum_{k=1}^{K} X^k - \sum_{k=1}^{K+1} X^k = -X^{K+1}.
\end{aligned}$$

Therefore,

$$\|(I - X)T_K - I\| = \|X^{K+1}\| \leq \|X\|^{K+1} \to 0$$

as $K \to \infty$. Therefore, $(I - X)T_K \to I$ as $K \to \infty$.

(c). Note that by Exercise 10.4 and 10.8(a),

$$\begin{aligned}
\|(I - X)T_K - (I - X)T\| &= \|(I - X)(T_K - T)\| \\
&\leq \|(I - X)\|\|T_K - T\| \\
&\leq (\|I\| + \|X\|)\|(T_K - T\| \leq 2\|T_K - T\|.
\end{aligned}$$

Since $\|T_K - T\| \to 0$ as $K \to \infty$, $(I - X)T_K \to (I - X)T$ as $K \to \infty$. But by part (b) $(I - X)T_K$ also converges to I. Therefore, $(I - X)T = I$. In order to show that $(I - X)^{-1} = T$ it only remains to show that $T(I - X) = I$. But this follows because for each K,

$$T_K(I - X) = \sum_{k=0}^{K} X^k - X \sum_{k=0}^{K} X^k = \sum_{k=0}^{K} X^k - \sum_{k=1}^{K+1} X^k = I - X^{K+1}.$$

But by the calculation in part (b), $(I - X)T_K = I - X^{K+1}$. Therefore, $T_K(I - X) = (I - X)T_K$ so that by taking limits as $K \to \infty$, $T(I - X) = (I - X)T = I$.