MATH 316 – HOMEWORK 7 – SOLUTIONS

Exercise 8.41.

Solution:

($\Longrightarrow$) Suppose that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$. We must show that the Cauchy condition is satisfied. By definition, given $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < \|\mathbf{x}-\mathbf{a}\| < \delta$ then $\|f(\mathbf{x})-L\| < \epsilon/2$. Therefore, if $\mathbf{x}, \mathbf{x}' \in B(\mathbf{a}, \delta)\setminus\{\mathbf{a}\}$ then $0 < \|\mathbf{x}-\mathbf{a}\| < \delta$ and $0 < \|\mathbf{x}'-\mathbf{a}\| < \delta$. Hence $\|f(\mathbf{x})-f(\mathbf{x}')\| \leq \|f(\mathbf{x})-L\|+\|f(\mathbf{x}')-L\| < \epsilon/2+\epsilon/2 = \epsilon$.

($\Longleftarrow$) Suppose that the Cauchy condition is satisfied. We must show that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists. We will do this by using the sequential criterion for limits, namely that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if and only if for every sequence $\mathbf{x}_n$ converging to $\mathbf{a}$ with $\mathbf{x}_n \neq \mathbf{a}$ for all n, $f(\mathbf{x}_n) \to L$. Let $\mathbf{x}_n$ be a sequence in the domain of f that converges to $\mathbf{a}$ but never equals $\mathbf{a}$. Given ϵ, there is a δ such that if $\mathbf{x}$, $\mathbf{x}' \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}$, then $\|f(\mathbf{x}) - f(\mathbf{x}')\| < \epsilon$. Choose N so large that if $n \geq N$ then $\|\mathbf{x}_n - \mathbf{a}\| < \delta$. Hence if n, $m \geq N$ then both $\mathbf{x}_n$ and $\mathbf{x}_m$ are in $B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}$ and so by the Cauchy criterion $\|f(\mathbf{x}_n) - f(\mathbf{x}_m)\| < \epsilon$. Therefore, $f(\mathbf{x}_n)$ is a Cauchy sequence and hence convergent, call the limit L.

It remains to show that all such sequences $f(\mathbf{x}_n)$ converge to L, and not to something else. Suppose that for some Cauchy sequence $\mathbf{y}_n$ in the domain of f, $\mathbf{y}_n \to \mathbf{a}$, $\mathbf{y}_n \neq \mathbf{a}$ for all n, and that $f(\mathbf{y}_n) \to M$. We will show that $M = L$. Given $\epsilon > 0$, let $\delta > 0$ be such that if $\mathbf{x}$, $\mathbf{x}' \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}$, then $\|f(\mathbf{x}) - f(\mathbf{x}')\| < \epsilon/3$, and let $\mathbf{x}_n$ be the sequence defined in the previous paragraph. Then since $\mathbf{x}_n \to \mathbf{a}$ and $\mathbf{y}_n \to \mathbf{a}$, there is an N such that if $n \geq N$ then $\mathbf{x}_n$, $\mathbf{y}_n \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}$, $\|f(\mathbf{x}_n) - L\| < \epsilon/3$, and $\|f(\mathbf{y}_n) - M\| < \epsilon/3$. Therefore

$$\|L - M\| \leq \|L - f(\mathbf{x}_n)\| + \|f(\mathbf{x}_n) - f(\mathbf{y}_n)\| + \|f(\mathbf{y}_n) - M\| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $L = M$.

An alternative way to prove this last part is to note that given $\epsilon > 0$ we can choose $\delta > 0$ such that if $\mathbf{x}$, $\mathbf{x}' \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}$, then $\|f(\mathbf{x}) - f(\mathbf{x}')\| < \epsilon/2$. Since $\mathbf{x}_n \to \mathbf{a}$ and $f(\mathbf{x}_n) \to L$, there is an N such that if $n \geq N$ then $\|f(\mathbf{x}_n)-L\| < \epsilon/2$, and $\mathbf{x}_n \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}$. Therefore, if $\mathbf{x} \in B(\mathbf{a}, \delta)$ then

$$\|f(\mathbf{x}) - L\| \leq \|f(\mathbf{x}) - f(\mathbf{x}_n)\| + \|f(\mathbf{x}_n) - L\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$.