MATH 316 – HOMEWORK 6 – SOLUTIONS

Exercise 8.41.

Solution:

(a) For $n = 1$ let $E_k = [k, \infty)$. Then $E_{k+1} \subseteq E_k$ for all k but $\bigcap_{k=1}^{\infty} E_k = \emptyset$. For $n > 1$, you can just set $E_k = [k, \infty) \times \cdots \times [k, \infty)$ where there are n terms in the product. Then the same conclusion holds.

(b) As in the hint, let $\mathbf{x}^{(k)} \in E_k$. Since $E_k \subseteq E_1$ for all k, $\mathbf{x}^{(k)} \in E_1$ for all k. Since E_1 is compact, the Bolzano-Weierstrass Theorem holds, so there is a subsequence $\mathbf{x}^{(k_j)}$ that converges to some $\mathbf{x} \in \mathbf{E}^n$. It remains to show that $\mathbf{x} \in \bigcap_{k=1}^{\infty} E_k$. Let $m \in \mathbf{N}$. We will show that $\mathbf{x} \in E_m$. Because $k_j \geq j$, we have that $k_j \geq m$ as soon as $j \geq m$. Therefore, if $j \geq m$, $x^{(k_j)} \in E_m$. In other words, the tail of the subsequence $x^{(k_j)}$ is in E_m. Since E_m is closed, it contains all of its cluster points, and it follows from this that $\mathbf{x} \in E_m$. (If this is not clear to you, you can reason as follows. If $\mathbf{x} \notin E_m$ then because $\mathbf{x}$ is the limit of a sequence of points in E_m, $\mathbf{x}$ satisfies the definition of cluster point of E_m. But this contradicts the assumption that E_m is closed. Hence $\mathbf{x} \in E_m$.)