MATH 316 – HOMEWORK 5 – SOLUTIONS

Exercise 8.26.

Solution:

($\Longrightarrow$) Suppose that S is open. We will show that $S = S^\circ$. By definition, $S^\circ \subseteq S$ so it remains only to show that $S \subseteq S^\circ$. Let $x \in S$. Since S is open there is an $r > 0$ such that $B(x, r) \subseteq S$. But this is exactly what it means for x to be an element of S°. Hence $x \in S^\circ$, $S \subseteq S^\circ$, and $S = S^\circ$.

($\Longleftarrow$) Suppose $S = S^\circ$. We must show that S is open. Let $x \in S$. Then $x \in S^\circ$ so there exists $r > 0$ such that $B(x, r) \subseteq S$. Therefore, S is open.

Exercise 8.31.

Solution:

In order to show that $\mathbf{Q}^n$ is dense in $\mathbf{E}^n$, we must show that $\overline{\mathbf{Q}^n} = \mathbf{E}^n$. Since $\mathbf{E}^n$ is the whole universe, $\overline{\mathbf{Q}^n} \subseteq \mathbf{E}^n$. We must show the opposite inclusion. Let $\mathbf{x} \in \mathbf{E}^n$. We must show that $\mathbf{x}$ is a cluster point of $\mathbf{Q}^n$. Let $\epsilon > 0$, and suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, for each j, there is an $r_j \in \mathbf{Q}$ such that $x_j < r_j < x_j + \epsilon/\sqrt{n}$, so that $0 < |x_j - r_j| < \epsilon/\sqrt{n}$. Let $\mathbf{r} = (r_1, r_2, \ldots, r_n)$. Then clearly $\mathbf{x} \neq \mathbf{r}$ so

$$0 < \|\mathbf{x} - \mathbf{r}\| = \left(\sum_{j=1}^{n} |x_j - r_j|^2 \right)^{1/2} < \left(\sum_{j=1}^{n} \frac{\epsilon^2}{n} \right)^{1/2} = \epsilon.$$

Hence $\mathbf{x}$ is a limit point of $\mathbf{Q}^n$, so that $\mathbf{E}^n \subseteq \overline{\mathbf{Q}^n}$ and hence $\mathbf{E}^n = \overline{\mathbf{Q}^n}$.