MATH 316 – HOMEWORK 3 – SOLUTIONS

Exercise 5.42.

Solution:

(a) Suppose that $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on D. This means that the partial sums form a uniformly Cauchy sequence. That is, given $\epsilon > 0$ there is an N such that if n, $m \geq N$ then $|\sum_{k=m}^{n} f_k(x)| < \epsilon$ for all $x \in D$. This is the same as saying that $\sup_{x\in D} |\sum_{k=m}^{n} f_k(x)| < \epsilon$. If we take $n = m$ then this becomes $\sup_{x\in D} |f_n(x)| = \|f_n\|_{sup} < \epsilon$. But this implies that $\lim \|f_n\|_{sup} = 0$.

The converse is false. If we let $f_k(x) = 1/k$ on $\mathbf{R}$ then $\sup_{x\in\mathbf{R}} |f_k(x)| = 1/k \to 0$ as $k \to \infty$ but clearly $\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} 1/k = \infty$.

2. Exercise 5.44.

Solution:

Showing $f \in \mathcal{C}^1(\mathbf{R})$ means showing that $f'(x)$ exists and is a continuous function at all points of $\mathbf{R}$. I claim that at each $x \in \mathbf{R}$

$$f'(x) = \sum_{k=1}^{\infty} \frac{d}{dx}\Big(\frac{\sin(kx)}{k^3}\Big) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}.$$

If this were the case, then by the Weierstrass M-test, $f'(x)$ would be continuous on $\mathbf{R}$. Specifically, $|\cos(kx)/k^2| \leq 1/k^2$ for all $x \in \mathbf{R}$, and since $1/k^2$ is summable the Weierstrass test says that the series $\sum_{k=1}^{\infty} \cos(kx)/k^2$ converges absolutely and uniformly on $\mathbf{R}$. Since each function $\cos(kx)/k^2$ is continuous on $\mathbf{R}$, each partial sum $\sum_{k=1}^{n} \cos(kx)/k^2$ is continuous and since the uniform limit of continuous functions is continuous, it follows that $\sum_{k=1}^{\infty} \cos(kx)/k^2$ defines a function continuous on $\mathbf{R}$.

To see that this formula holds for each x, we will use Theorem 5.5.1 (iii). First note that if we can show that the formula for $f'(x)$ holds for all $x \in [-R, R]$ for all $R > 0$ then that will imply that it holds for all $x \in \mathbf{R}$. In this case, $f_k(x) = \sin(kx)/k^3$. Note that each f_k is $\mathcal{C}^1$ on $[-R, R]$, that is, each f_k is continuously differentiable on $[-R, R]$. Also, note that if $x = 0$ then since $f_k(0) = 0$ for all k, $\sum_{k=1}^{\infty} f_k(0)$ converges. Finally, we have shown in the above paragraph that the series

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges uniformly on $\mathbf{R}$ and hence on $[-R, R]$ for all $R > 0$. Therefore, by Theorem 5.5.1(iii),

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

as required.