MATH 316 – HOMEWORK 2 – SOLUTIONS

Exercise 5.17.

Solution:

(a) Suppose that $\limsup(x_k)^{1/k} = L < 1$, and let $L < r < 1$ (taking $r = (L+1)/2$ will do). Since

$$\limsup(x_k)^{1/k} = \lim_n \sup_{k\geq n}(x_k)^{1/k} = L,$$

there is an N such that if $n \geq N$ then $\sup_{k\geq n}(x_k)^{1/k} \leq r$ (this comes from the definition of limit for $\epsilon = L - r > 0$). In particular, $\sup_{k\geq N}(x_k)^{1/k} \leq r$ so that $(x_k)^{1/k} \leq r$ for all $k \geq N$. Therefore, for $n \geq N$, $0 \leq x_k \leq r^k$. Since $0 < r < 1$, r^k is summable. Therefore by Theorem 5.2.1, x_k is summable.

(b) Suppose that $\limsup(x_k)^{1/k} = L > 1$. Since

$$\limsup(x_k)^{1/k} = \lim_n \sup_{k\geq n}(x_k)^{1/k} = L,$$

and since $\sup_{k\geq n}(x_k)^{1/k}$ is a decreasing sequence, $\sup_{k\geq n}(x_k)^{1/k} \geq L > 1$ for all n. This means that for every N, there is an $n \geq N$ such that $(x_n)^{1/n} > 1$. Therefore, we can find a subsequence x_{n_k} of x_n with the property that $x_{n_k} \geq 1^{n_k} = 1$ for all k. Hence, x_n does not converge to 0 as $n \to \infty$ and so by the n-th Term Test, the sequence x_n is not summable.

2. Exercise 5.22.

Solution:

Note first that by the Mean Value Theorem and the fact that $|f'(x)| < 1$ for all $x \in (0,1]$, given any x_1 and $x_2 \in (0,1]$, $|f(x_1) - f(x_2)| = |f'(\mu)||x_1 - x_2| < |x_1 - x_2|$.

(a). Following the hint, we note that by above,

$$\left|f\left(\frac{1}{n+1}\right) - f\left(\frac{1}{n}\right)\right| \leq \left|\left(\frac{1}{n+1}\right) - \left(\frac{1}{n}\right)\right| = \left|\frac{1}{n(n+1)}\right| \leq \frac{1}{n^2}.$$

By the p-series test, $1/n^2$ is summable, hence so is $|f(1/(n+1)) - f(1/n)|$, that is, $\sum_{n=1}^{\infty} f(1/(n+1)) - f(1/n)$ is absolutely convergent. Since it is absolutely convergent, it is convergent and it is easy to write down a formula for the n-th partial sum, viz.,

$$s_n = \sum_{k=1}^{n} f\left(\frac{1}{k+1}\right) - f\left(\frac{1}{k}\right) = f\left(\frac{1}{n+1}\right) - f(1).$$

Therefore,

$$\lim f(1/n) = \lim f(1/(n+1)) = f(1) + \lim s_n = L.$$

(b). We have to show that given $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in (0, \delta)$ then $|f(x) - L| < \epsilon$. Having just one sequence x_n with $x_n \to 0$ and with $f(x_n) \to 0$ (which is part (a) with $x_n = 1/n$) is not enough to guarantee this. Given $\epsilon > 0$, choose $\delta_0 > 0$, so that if $n \geq 1/\delta_0$ then $|f(1/n) - L| < \epsilon/2$. Let $\delta < \max\{\delta_0, \epsilon/2\}$. If $x \in (0, \delta)$, let n be such that $1/n \in (0, \delta)$. Then also, $n \geq 1/\delta_0$ and once again by the above calculation,

$$|f(x)-L| \leq |f(x)-f(1/n)|+|f(1/n)-L| \leq |x-1/n|+|f(1/n)-L| < \epsilon/2+\epsilon/2 = \epsilon.$$