MATH 316 – HOMEWORK 1 – SOLUTIONS

Exercise 5.7.

Solution:

(a) We must show that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < \epsilon$. Let $\epsilon > 0$ and choose $\delta = \epsilon/r$. If $|x_1 - x_2| < \delta = \epsilon/r$, $r|x_1 - x_2| < \epsilon$. But by hypothesis, $|f(x_1) - f(x_2)| \le r|x_1 - x_2| < \epsilon$.

(b) This can be done formally by induction. If $n = 1$ then $|x_2 - x_1| = |f(x_1) - f(x_0)| \le r|x_1 - x_0|$ as required. Let n be given and suppose that $|x_{n+1} - x_n| \le r^n|x_1 - x_0|$. We must show that $|x_{n+2} - x_{n+1}| \le r^{n+1}|x_1 - x_0|$. By definition,

$$|x_{n+2} - x_{n+1}| = |f(x_{n+1}) - f(x_n)| \le r|x_{n+1} - x_n| \le r \cdot r^n|x_1 - x_0| = r^{n+1}|x_1 - x_0|$$

as required.

(c) We must show that given $\epsilon > 0$ there is an N such that if $n, m \ge N$ then $|x_n - x_m| < \epsilon$. First of all, we can assume with out loss of generality that $x_1 \neq x_0$ because if $x_1 = x_0$ then for all n, $x_{n+1} = x_n$ and the sequence is constant, which is of course Cauchy. Assuming without loss of generality that $n > m$, we can write

$$\begin{aligned}
|x_n - x_m| &= \Big|\sum_{j=m}^{n-1}(x_{j+1} - x_j)\Big| \\
&\le \sum_{j=m}^{n-1}|x_{j+1} - x_j| \\
&\le |x_1 - x_0| \sum_{j=m}^{n-1} r^j \\
&= r^m\,|x_1 - x_0| \sum_{j=0}^{n-m-1} r^j \\
&\le r^m\,|x_1 - x_0| \sum_{j=0}^{\infty} r^j \\
&= r^m\,\frac{|x_1 - x_0|}{1 - r}.
\end{aligned}$$

Given $\epsilon > 0$, choose N so large that if $m \ge N$ then $r^m < \epsilon\,((1 - r)/|x_1 - x_0|)$. This can be done since $r^m \to 0$ as $m \to \infty$. Therefore, if $n, m \ge N$ then in particular $m \ge N$ so that by the above calculation, $|x_n - x_m| < \epsilon$ as required.

(d) Since the sequence x_n is Cauchy it converges so let $p = \lim x_n$. Since f is continuous on $\mathbf{R}$, $f(p) = \lim f(x_n) = \lim x_{n+1} = p$. If it is not clear to you that $\lim x_{n+1} = \lim x_n$ you should show it by going to the definition of limit.

(e) Suppose that $f(p) = f(q)$. By our assumption on f, $|p-q| = |f(p) - f(q)| < r|p-q|$. If $p \neq q$, we can divide both sides of the inequality by $|p-q|$ and obtain $1 < r$ contradicting our assumption that $r < 1$. Therefore, $p = q$.

2. Exercise 5.8.

Solution:

(a). Assume that $L > 0$. Since $\lim(x_n/y_n) = L > 0$, there is an N such that if $n \geq N$ then $|(x_n/y_n) - L| < L/2$. This implies that $L/2 < |x_n/y_n| < 3L/2$ (just apply the definition of $|a| < b$), or that $(L/2)|y_n| < |x_n| < (3L/2)|y_n|$. If y is summable, then since $y_k > 0$, $|y_k|$ is summable and by Exercise 5.6(b), $(3L/2)|y_n|$ is summable. By Theorem 5.2.1, $|x_k| = x_k$ is also summable. If x is summable, then since $x_k > 0$, $|x_k|$ is summable and by Exercise 5.6(b), $(2/L)|y_n|$ is summable. Since $|y_k| \leq (2/L)|x_k|$, by Theorem 5.2.1, $|y_k| = y_k$ is also summable.

(b). Assume that $L \geq 0$. If $L > 0$ then part (a) implies that if y is summable then x is summable. If $L = 0$, then there is an N such that if $n \geq N$ then $|x_n/y_n| < 1$, or $|x_n| < |y_n|$. Since both x_k and y_k are non-negative, $0 \leq x_k \leq y_k$ for all $k \geq N$. By Theorem 5.2.1, if y is summable then x is summable.