MATH 316 — MIDTERM EXAM — SOLUTIONS

1. Let x_k be absolutely summable. We must show that $\sum_{k=1}^{\infty} x_k$ converges. We will show that its partial sums are Cauchy. Let $S_n = \sum_{k=1}^{n} x_k$, and let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} |x_k|$ converges there is an N such that $n, m \geq N$ implies that $\sum_{k=m}^{n} |x_k| < \varepsilon$. For this N,

$$|S_n - S_m| = \left| \sum_{k=m}^{n-1} x_k \right| \leq \sum_{k=m}^{n-1} |x_k| < \varepsilon.$$

Hence S_n converges and x_k is summable.

2. Let $x_k = (-1)^k$. Then $S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} (-1)^k$

$$= \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Hence S_n is bounded.

However the sequence S_n does not converge because always $|S_{n+1} - S_n| = 1$, so S_n is not Cauchy.

3. ($\Rightarrow$) Suppose that x_k is summable. This means that the sequence s_n of partial sums is convergent. But we know that convergent sequences are bounded. Hence s_n is bounded.

($\Leftarrow$) Suppose that s_n is bounded. We must show it is convergent. Since $x_k \geq 0$ all k, s_n is a non-decreasing sequence i.e. $S_{n+1} \geq S_n$ for all n. Since S_n is bounded then $S = \sup_n S_n$ exists (by the Completeness of $\mathbb{R}$). We will show that $S_n \to S$. Let $\varepsilon > 0$. Since $S = \sup_n S_n$, there is an N such that $S_N > S - \varepsilon$, or $S - S_N < \varepsilon$. Since $S_N \leq S$ for all N, $|S - S_N| < \varepsilon$. Since S_n is increasing, $S - S_n$ is decreasing so if $n \geq N$, $|S - S_n| \leq |S - S_N| < \varepsilon$. Hence $S_n \to S$.

4. In order to show $\sum f_k$ converges uniformly on D we must show that given $\varepsilon > 0$ there is an N such that if $m, n \geq N$ then $\|s_n - s_m\|_{sup} = \|\sum_{k=m+1}^{n} f_k\|_{sup} < \varepsilon$.

But since M_k is summable, there is an N such that $n, m \geq N$ implies that $\sum_{k=m+1}^{n} M_k < \varepsilon$. Hence if $n, m \geq N$,

$$\|\sum_{k=m+1}^{n} f_k\|_{sup} \leq \sum_{k=m+1}^{n} \|f_k\|_{sup} = \sum_{k=m+1}^{n} M_k < \varepsilon.$$

(Here we have used the fact that $\sup_{x \in D} |f(x) + g(x)| \leq \sup_{x \in D} |f(x)| + \sup_{x \in D} |g(x)|$)

5. (A) Let $\{\mathcal{O}_\alpha\}$ be a collection of open sets in $\mathbb{R}^n$ and let $\mathcal{O} = \bigcup_\alpha \mathcal{O}_\alpha$. We must show $\mathcal{O}$ is open. Let $\vec{x} \in \mathcal{O}$ then for some α_0, $\vec{x} \in \mathcal{O}_{\alpha_0}$. Since $\mathcal{O}_{\alpha_0}$ is open there is an $\varepsilon > 0$ such that $B(\vec{x}, \varepsilon) \subseteq \mathcal{O}_{\alpha_0}$. But since $\mathcal{O}_{\alpha_0} \subseteq \mathcal{O}$, $B(\vec{x}, \varepsilon) \subseteq \mathcal{O}$ and $\mathcal{O}$ is open.

(B) Let $\{\mathcal{O}_n\}_{n=1}^N$ be open sets and let $\mathcal{O} = \bigcap_{n=1}^N \mathcal{O}_n$. We must show $\mathcal{O}$ is open.

Let $\vec{x} \in \mathcal{O}$. Then $\vec{x} \in \mathcal{O}_n$ for each n so there is an $\varepsilon_n > 0$ such that $B(\vec{x}, \varepsilon_n) \subseteq \mathcal{O}_n$ for each n. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. We will show $B(\vec{x}, \varepsilon) \subseteq \mathcal{O}$. If $\vec{y} \in B(\vec{x}, \varepsilon)$ then since $\varepsilon_n \geq \varepsilon$ all n, $\vec{y} \in B(\vec{x}, \varepsilon_n) \subseteq \mathcal{O}_n$. Hence $\vec{y} \in \bigcap_{n=1} \mathcal{O}_n = \mathcal{O}$, and so $B(\vec{x}, \varepsilon) \subseteq \mathcal{O}$. Hence $\mathcal{O}$ is open.