MATH 316 – SPRING 2013 – MIDTERM EXAM

Answer all of the following questions on the sheets provided. You are to show all work but try to be as neat as possible. It is required only that your solutions be legible and reasonably well organized. You may turn in your scratch work if you like but make sure that it is clear to me what is your final solution and what is scratch work. You may use the definitions and results provided, and other basic facts as required.

Let x_k be a sequence of real numbers. We say that x_k is *summable* if the series $\sum_{k=1}^{\infty} x_k$ converges, and that x_k is *absolutely summable* if the series $\sum_{k=1}^{\infty} |x_k|$ converges. The sequence of *partial sums* of the series $\sum_{k=1}^{\infty} x_k$ is the sequence s_n defined by $s_n = \sum_{k=1}^{n} x_k$. A series $\sum_{k=1}^{\infty} x_k$ *converges* if the sequence of its partial sums converges. The *Cauchy criterion* says that a sequence s_n of real numbers converges if and only if it is Cauchy, that is, if for every $\epsilon > 0$ there is an N such that if $n,\, m \geq N$ then $|s_n - s_m| < \epsilon$.

Let f_k be a sequence of functions defined on a subset $D \subseteq \mathbf{R}$. We say that f_k *converges pointwise* if there is a function f on D such that for each $x \in D$, the sequence of numbers $f_k(x)$ converges to $f(x)$. We say that f_k *converges uniformly* if there is a function f on D such that $\lim_{k\to\infty} \|f_k - f\|_{sup} = 0$, where for a function h defined on D, $\|h\|_{sup} = \sup\{|h(x)| \colon x \in D\}$. The sequence f_k is *uniformly Cauchy* on D if for every $\epsilon > 0$ there is an N such that if $n,\, m \geq N$ then $\|f_n - f_m\|_{sup} < \epsilon$. The sequence f_k is uniformly Cauchy if and only if it is uniformly convergent.

The *open ball* centered at $\mathbf{a} \in \mathbf{R}^n$ with radius r is the set $B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbf{R}^n \colon \|\mathbf{x} - \mathbf{a}\| < r\}$, where $\|\mathbf{x}\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, $\mathbf{x} = (x_1,\, x_2,\, \ldots,\, x_n)$ is the usual Euclidean norm. A set $\mathcal{O} \subseteq \mathbf{R}^n$ is *open* if for each $\mathbf{x} \in \mathcal{O}$, there is an $\epsilon > 0$ such that $B(\mathbf{x}, \epsilon) \subseteq \mathcal{O}$.

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1. (10 pts.) Prove that if the sequence x_k is absolutely summable, then it is summable.

2. (10 pts.) Give an example of a sequence x_k that is *not* summable but such that the sequence of its partial sums is bounded. Be sure to prove your assertions.

3. (10 pts.) Suppose that x_k is a nonnegative sequence, i.e., $x_k \geq 0$. Prove that x_k is summable if and only if the sequence of its partial sums is bounded.

4. (10 pts.) Let f_k be a sequence of functions defined on a subset $D \subseteq \mathbf{R}$. Prove the Weierstrass M-test, that is, prove that if for each k, $\|f_k\|_{sup} = M_k$, and M_k is a summable sequence, then the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on D.

5. (10 pts.) Prove that the arbitrary union of open sets in $\mathbf{R}^n$ is open, and that the finite intersection of open sets in $\mathbf{R}^n$ is open.