11.1 Definition of the Integral.

A. Integration on the line.

1. Definition 1. A *partition* P of an interval $[a, b]$ is a finite set $P = \{x_0, x_1, \dots, x_n\}$ where
$$a = x_0 < x_1 < \cdots < x_n = b.$$
Given a bounded function f defined on $[a, b]$, define
$$M_k = \sup\{f(x):\ x \in [x_{k-1}, x_k]\},$$
and
$$m_k = \inf\{ f(x):\ x \in [x_{k-1}, x_k]\}.$$

Define the *upper and lower sums* of f by
$$U(f, P) = \sum_{k=1}^{n} M_k\,(x_k - x_{k-1})$$
$$L(f, P) = \sum_{k=1}^{n} m_k\,(x_k - x_{k-1})$$
If
$$\sup_P L(f, P) = \inf_P U(f, P).$$
then $\int_a^b f(x)dx$ is this common number and f is said to be *Riemann integrable on* $[a, b]$ or $f \in \mathcal{R}[a, b]$.

B. Integration on $\mathbb{E}^n$.

1. What replaces the interval $[a, b]$?

<u>Definition 2</u>. A *closed rectangular block* (or just a *rectangle*) $B = [\mathbb{a}, \mathbb{b}] \subseteq \mathbb{E}^n$ is defined by

$$B = [\mathbb{a}, \mathbb{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

2. What replaces a partition?

Definition 3. Given a rectangle $[\mathbb{a}, \mathbb{b}]$ in $\mathbb{E}^n$, a *partition* $\mathcal{P}$ of $[\mathbb{a}, \mathbb{b}]$ is the Cartesian product

$$\mathcal{P} = P_1 \times \cdots \times P_n$$

where each P_i is a partition of $[a_i, b_i]$. A *grid* is a collection of rectangles of the form $I_1 \times \cdots \times I_n$ where each I_j is a subinterval of the partition P_j, that is, $I_j = [x^j_{k-1}, x^j_k]$.

3. We also need to specify the notion of volume.

<u>Definition</u> 4. Let $S \subseteq \mathbb{E}^n$ and suppose that $S \subseteq B$ for some rectangle B. Let $\mathcal{P}$ be a grid on B, and define

$$V(S,\mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \cap \bar{S} \neq \emptyset}} vol(B_j);$$

$$v(S,\mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \subseteq S^\circ}} vol(B_j)$$

where $\bar{S}$ is the closure and S° the interior of S.

The *inner volume* of S is

$$vol(S) = \sup\{ v(S,\mathcal{P}) : \mathcal{P}\ a\ grid\ on\ B\}$$

and the *outer volume* is

$$Vol(S) = \inf\{ V(S,\mathcal{P}) : \mathcal{P}\ a\ grid\ on\ B\}$$

If $vol(S) = Vol(S)$ then S is called a *Jordan region* and the common value is the *volume* or *Jordan content* of S and is denoted $v(S)$.

4. Theorem. A bounded set $S \subseteq \mathbb{E}^n$ is a Jordan region if and only if $Vol(\partial S) = 0$ where $\partial S = \bar{S} \setminus S^\circ$ is the boundary of S.

Such a set is called a *Jordan null set*.

C. Integration over Jordan Regions.

1. Definition 5. Let $S \subseteq \mathbb{E}^n$ be a Jordan region, $f: S \to \mathbb{E}^1$ be bounded, and suppose that $S \subseteq B$ for some rectangle B. Extend f to a function on B by letting $f(x) = 0$ if $x \in B \setminus S$.

   The *upper sum* of f with respect to a grid $\mathcal{P}$ of B is

$$U(f, \mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \cap E \neq \emptyset}} M_j \, \nu(B_j)$$

   where $M_j = \sup\{f(\mathbb{x}) : \mathbb{x} \in B_j\}$ and the *lower sum* by

$$L(f, \mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \cap E \neq \emptyset}} m_j \, \nu(B_j)$$

   where $m_j = \inf\{f(\mathbb{x}) : \mathbb{x} \in B_j\}$.

2. <u>Definition</u> 6. The upper and lower integrals of f over S are given by

$$\overline{\int_S} f = \sup\{L(f, \mathcal{P}) \colon \mathcal{P} \ a \ grid \ on \ B\}$$

and

$$\underline{\int_S} f = \inf\{U(f, \mathcal{P}) \colon \mathcal{P} \ a \ grid \ on \ B\}$$

if $\overline{\int_S f} = \underline{\int_S f}$ then the common value is the Riemann integral of f on S and is denoted $\int_S f$.

3. <u>Remark</u>.
   a. The value of the integral over S is independent of the choice of rectangle B containing S.

   b. What is the point of defining the integral this way, i.e., by enclosing S in a rectangle?